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Topic

- Simple review of Fourier transform

- 2-D Fourier transform

Simple review of Fourier transform

The Fourier Transform

- The Fourier transform converts a signal from spatial domain to the frequency domain.

$$f(t) \longrightarrow \boxed{\text{Fourier Transform}} \longrightarrow F(s)$$

- The Continuous Fourier Transform of a one-dimensional continuous function $f(t)$ is defined as

$$\mathbf{F}\{f(t)\} = F(s) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi st} dt$$

The Fourier Transform

- The inverse Fourier transform

$$\mathbf{F}^{-1}\{F(s)\} = \int_{-\infty}^{\infty} F(s) e^{j2\pi st} ds$$

$$F(s) \longrightarrow \boxed{\text{Inverse Fourier Transform}} \longrightarrow f(t)$$

The Discrete Fourier Transform

- If $\{f_i\}$ is a sequence of length N, then its *discrete Fourier transform* (DFT) is given by

$$F_n = \frac{1}{N}\sum_{i=0}^{N-1} f_i e^{-j2\pi\frac{n}{N}i}, \quad n = 0,1,\cdots N-1$$

- And the inverse DFT is given by

$$f_i = \sum_{n=0}^{N-1} F_n e^{-j2\pi\frac{i}{N}n}, \quad i = 0,1,\cdots,N-1$$

Where $0 \le i,n \le N-1$ are indices.

Properties of the Fourier Transform

- The Addition Theorem
  - If $\mathbf{F}\{f(t)\}=F(s)$ and $\mathbf{F}\{g(t)\}=G(s)$, then

$$\mathbf{F}\{f(t)+g(t)\}=F(s)+G(s)$$

  - And take it as an axiom that for any real number *c*

$$\mathbf{F}\{cf(t)\}=cF(s)$$

  - This implies that Fourier transform is a linear transform.

The Shift Theorem

- Time shift:

$$\mathbf{F}\{f(t-a)\} = e^{-j2\pi as}F(s)$$

- Frequency shift:

$$\mathbf{F}\{f(t)e^{j2\pi s_0 t/N}\} = F(s-s_0)$$

When $s_0 = N/2$

$$\mathbf{F}\{f(t)(-1)^t\} = F(s-N/2)$$

The Convolution Theorem

- The Fourier transform of the convolution of two functions is the product of the Fourier transforms of these two functions

$$\mathbf{F}\{f(t) * g(t)\} = F(s)G(s)$$

The Differentiation Theorem

- Another property of the FT is the conversion of derivative in time domain into a simple multiplication process in the frequency domain. This property is used to convert differential equations in time into a set of simple linear equations in frequency domain and solve multidimensional differential equations using simple linear algebra

$$\mathbf{F}\{f(t) * g(t)\} = F(s)G(s)$$

The Scaling Property

- An extremely useful property of the FT is the way time and frequency domains are inversely scaled. Specifically, assume that the FT of a signal g(t) is given as G(s), the scaling property states that, for a signal defined as $g_1(t) = g(\alpha t)$ with $\alpha > 1$, we can easily calculate the FT using $G(f)$ as fellow:

$$FT\{g_1(t)\} = G_1(f) = \frac{1}{a} G\left(\frac{f}{a}\right)$$

2-D Fourier Transform

The 2-D Continuous Fourier Transform

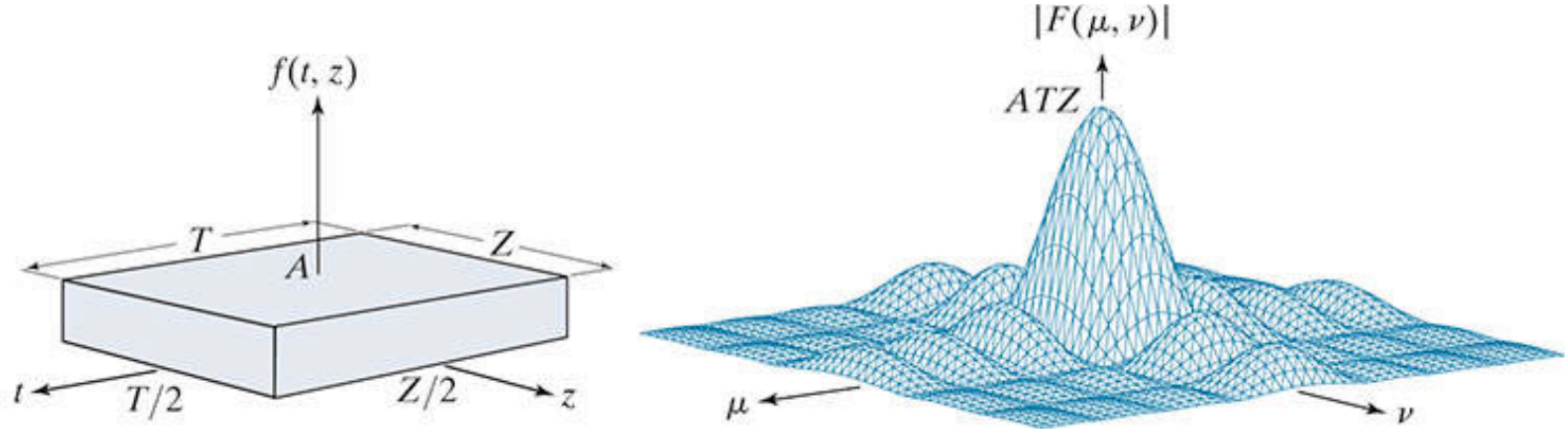
- The discrete Fourier transform of an image of size M*N is given by

$$F(\mu, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, z) e^{-j2\pi(\mu t + vz)} \, dt \, dz$$

- Inverse Fourier transform

$$f(t, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\mu, v) \, e^{j2\pi(\mu t + vz)} \, d\mu \, dv$$

The 2-D Continuous Fourier Transform

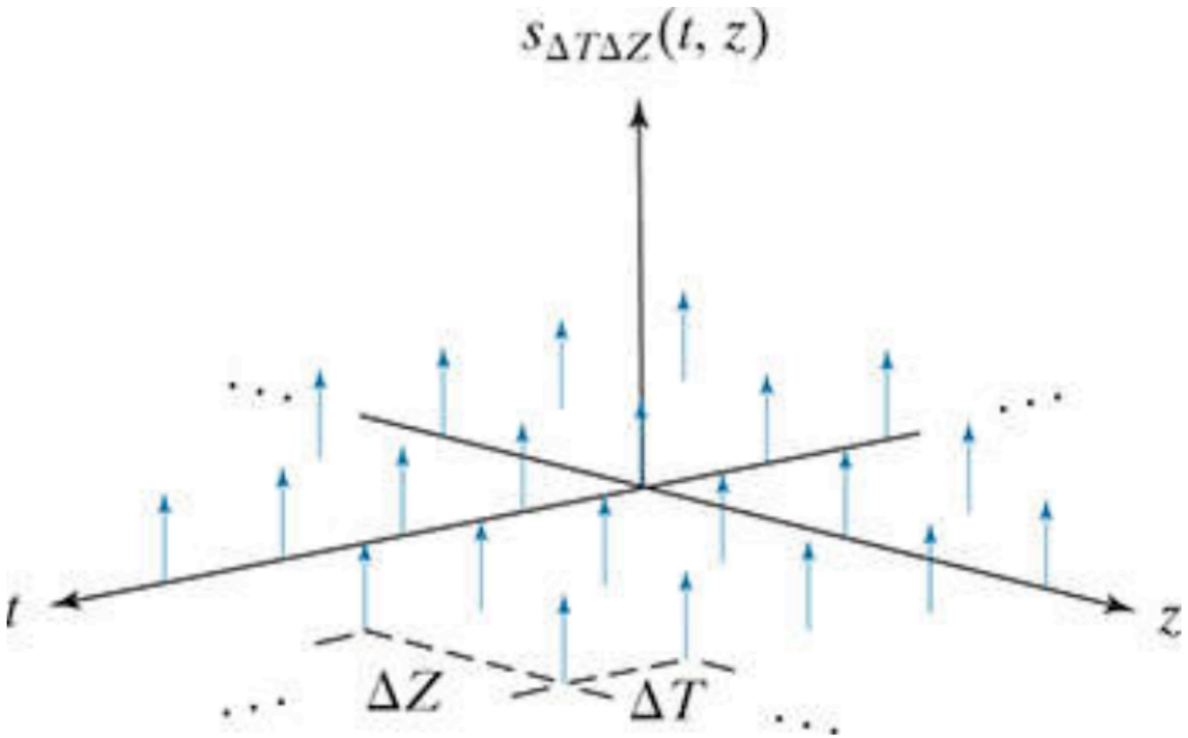


(a)

(b)

(a) The 2-D equivalent of the 1-D box function. (b) The spectrum of (a). The box is longer along the t-axis, so the spectrum is more contracted along the μ-axis.

2-D Sampling Theorem



2-D impulse train:

$$s_{\Delta T \Delta Z}(t,z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(t - m\Delta T, z - n\Delta Z)$$

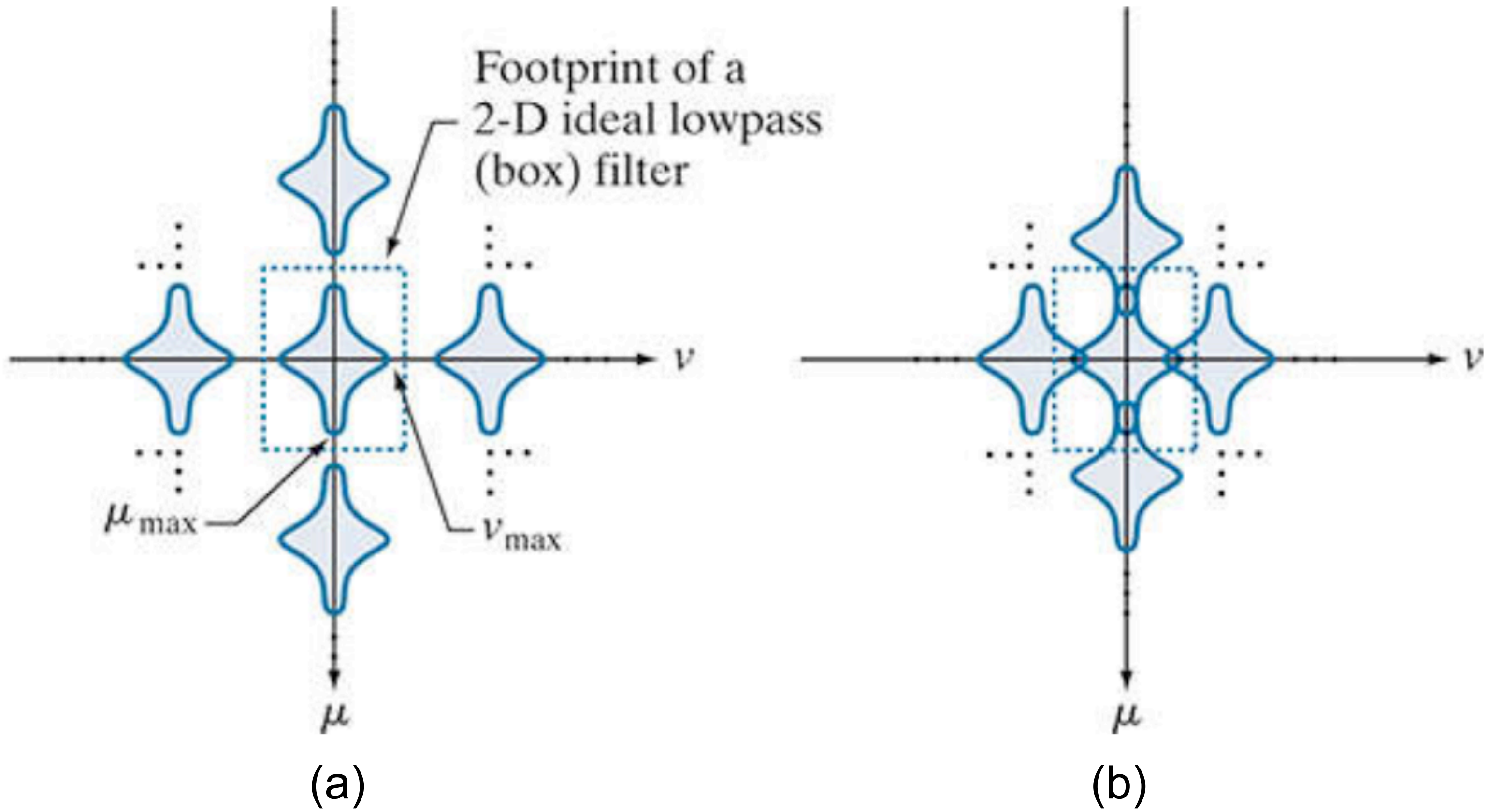
Limited band:

$$F(\mu, v) = 0 \quad \text{for } |\mu| \geq \mu_{\max} \quad \text{and } |v| \geq v_{\max}$$

Recoverable sampling rate:

$$\frac{1}{\Delta T} > 2\mu_{\max} \qquad \frac{1}{\Delta Z} > 2v_{\max}$$

2-D Sampling Theorem



Two-dimensional Fourier transforms of (a) an over-sampled, and (b) an under-sampled, band-limited function.

(a)

(b)

Illustration of aliasiing on resampled images.

(a) A digital image of size 772 × 548 pixels with visually negligible aliasing.

(b) Result of resizing the image to 33% of its original size by pixel deletion and then restoring it to its original size by pixel replication. Aliasing is clearly visible.

The Two-Dimensional DFT

- The discrete Fourier transform of an image of size M*N is given by

$$F(u,v)=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

- Inverse Fourier transform

$$f(x,y)=\frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$$

The Two-Dimensional DFT

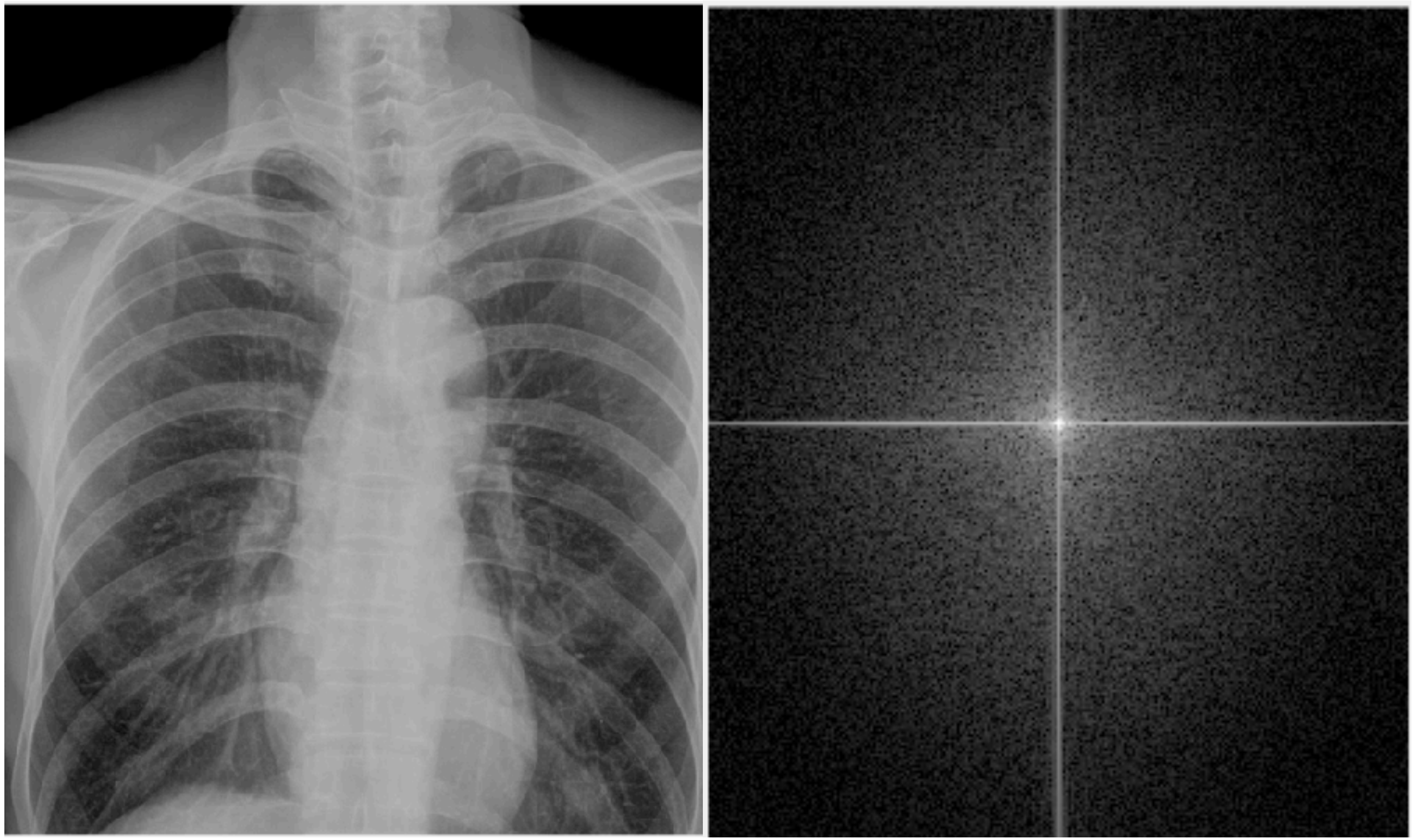
- What's the value of the transform at (u, v)=(0, 0)?

$$F(0,0) = MN\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)$$

$$= MN\bar{f}$$

- MN times of the average of f(x, y). Because the proportionality constant MN usually is large, $|F(0,0)|$ typically is the largest component of the spectrum by a factor that can be several orders of magnitude larger than other terms. Because frequency components u and v are zero at the origin, $|F(0,0)|$ sometimes is called the *dc component* of the transform.

DFT on Common Medical Images

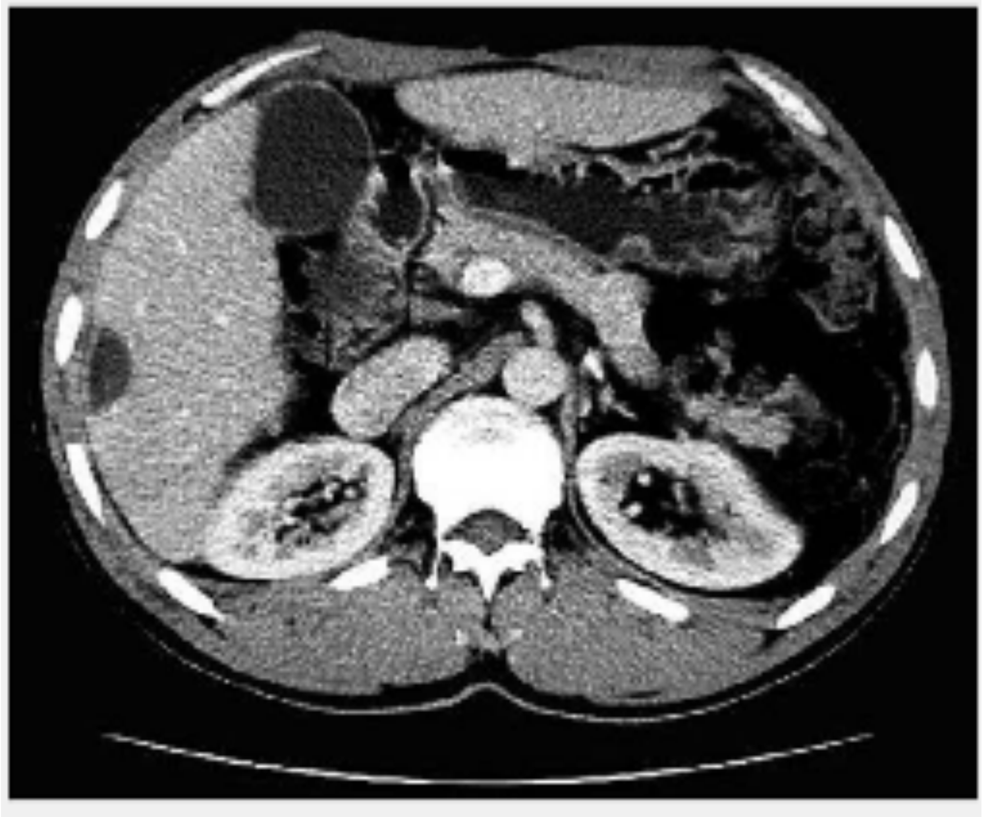


(a) (b)

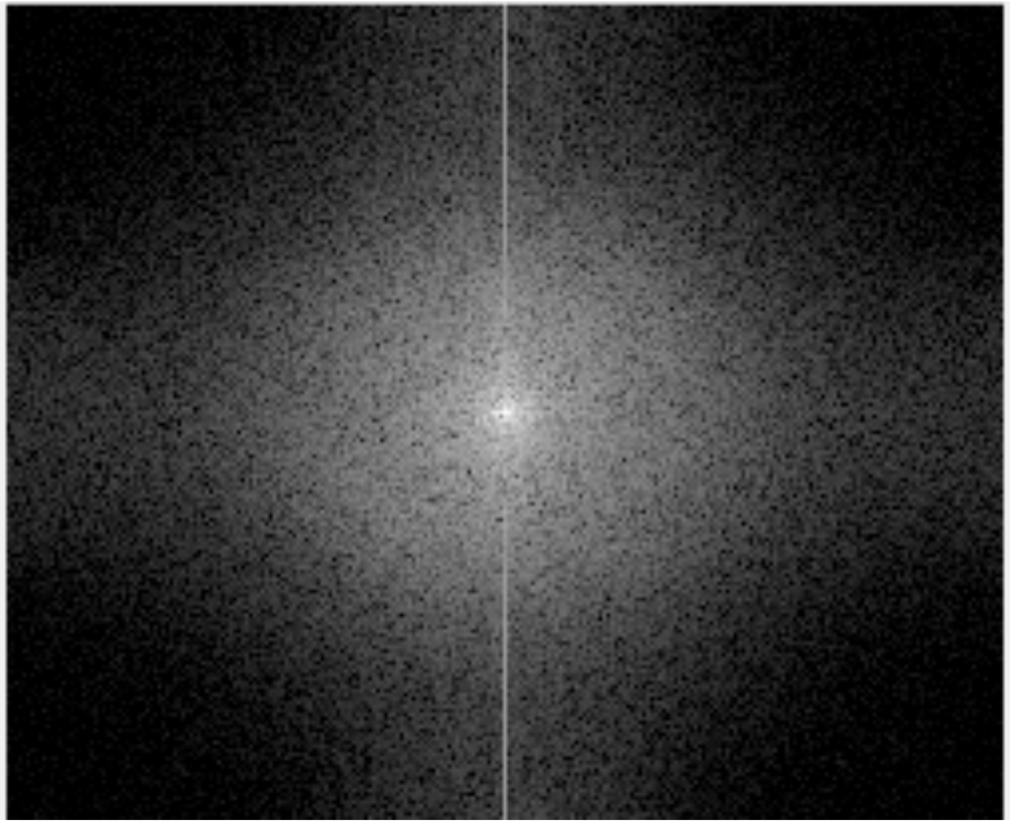
(a) The X-ray image

(b) 2-D DFT of (a)

DFT on Medical Images



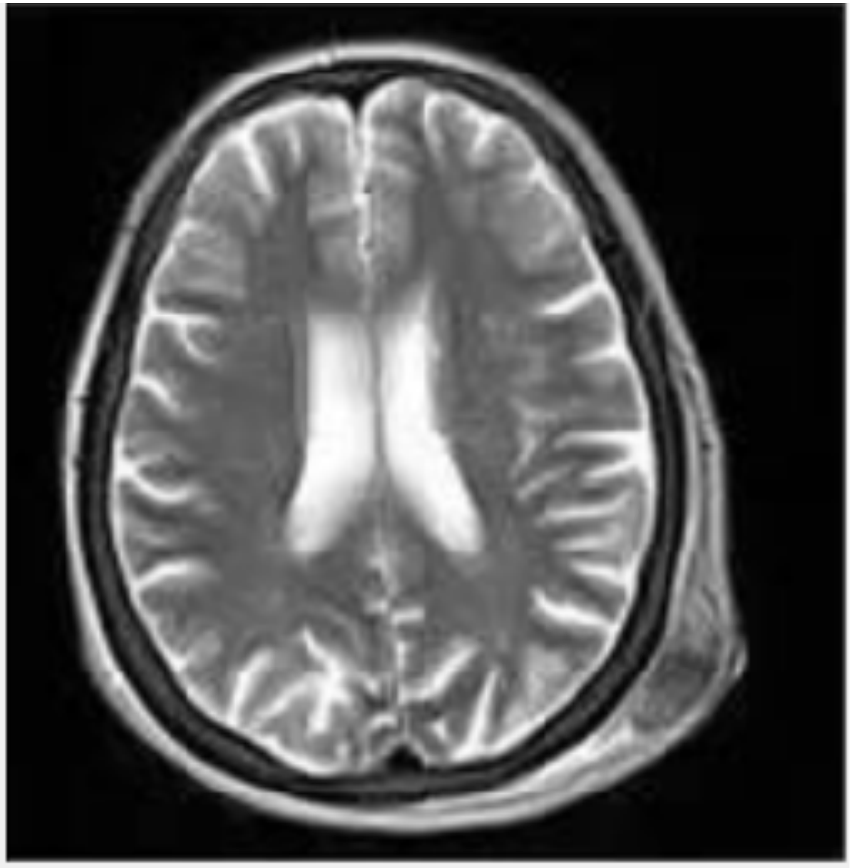
(a)



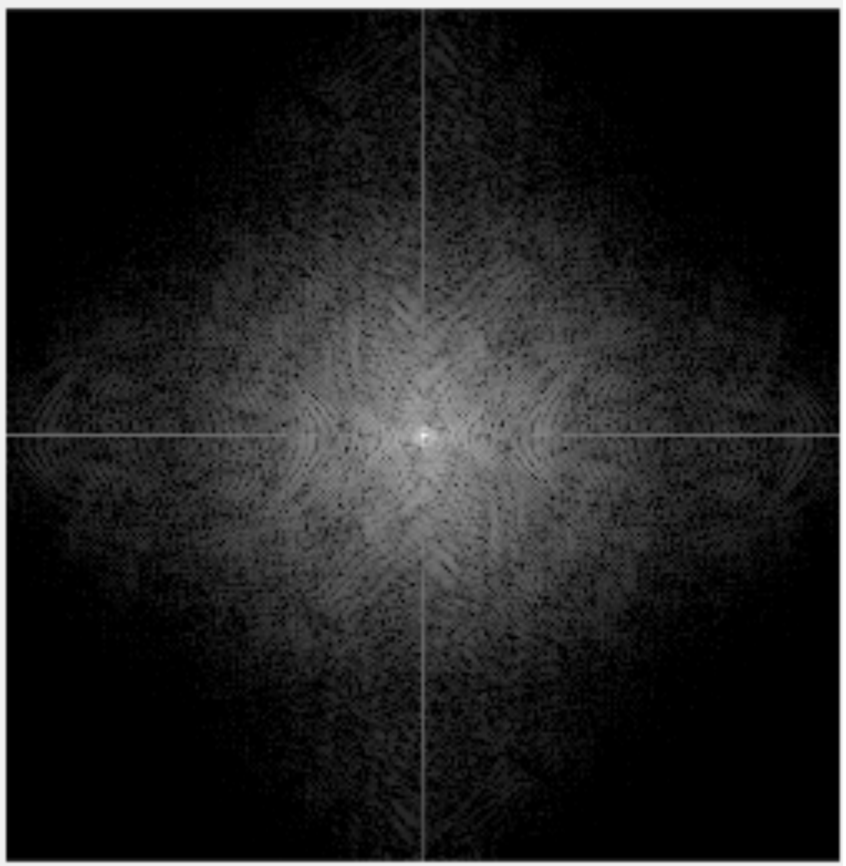
(b)

(a) CT image. (b)2-D DFT of (a)

DFT on Medical Images



(a)

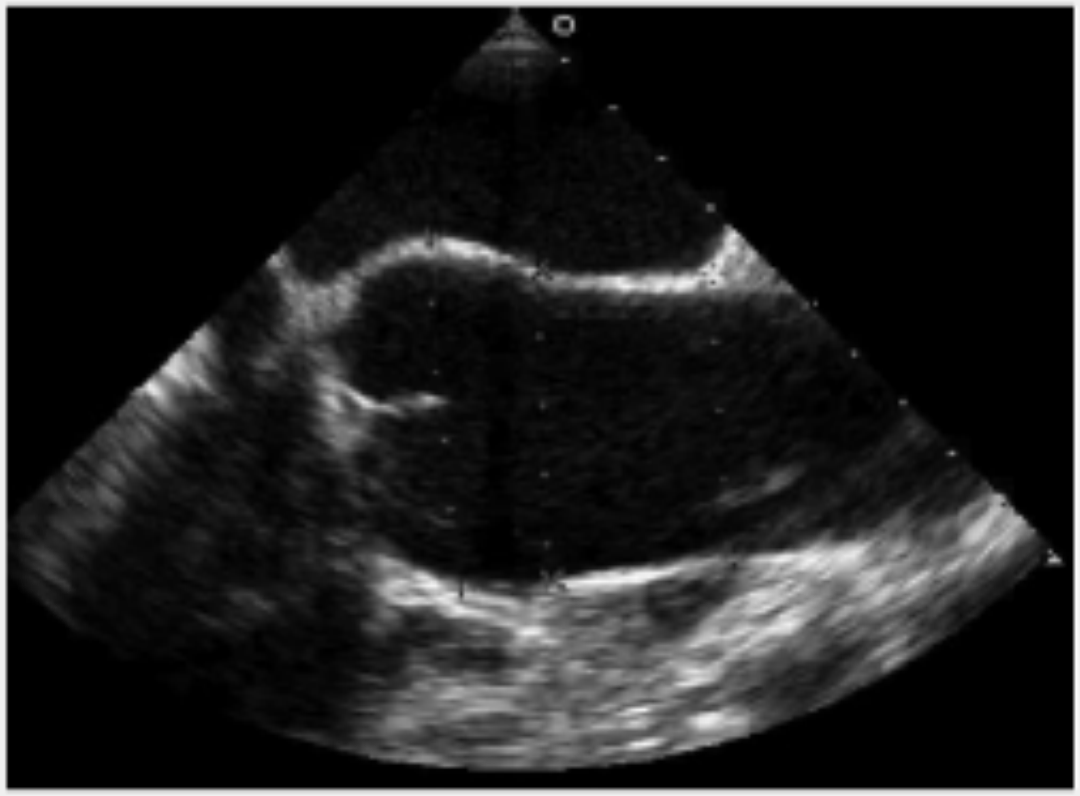


(b)

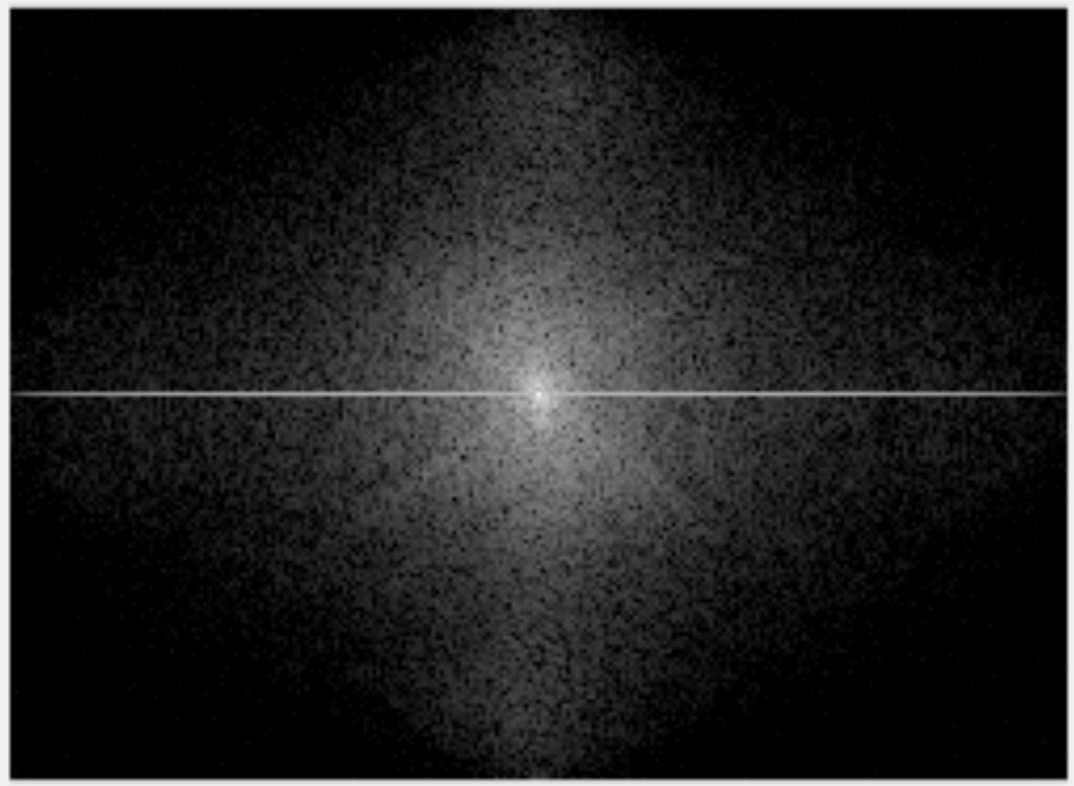
(a) The MRI image

(b) 2-D DFT of (a)

DFT on Medical Images



(a)



(b)

(a) The ultrasound image. (b) 2-D DFT of (a)

Properties of the 2-D DFT and IDFT

- Relationships between spatial and frequency intervals

$$\Delta u = \frac{1}{M \Delta T} \qquad \Delta v = \frac{1}{N \Delta Z}$$

- Translation and rotation

$$f(x, y) e^{j2\pi(u_0 x/M + v_0 y/N)} \Leftrightarrow F(u - u_0, v - v_0)$$

$$f(x - x_0, y - y_0) \Leftrightarrow F(u, v)\, e^{-j2\pi(x_0 u/M + y_0 v/N)}$$

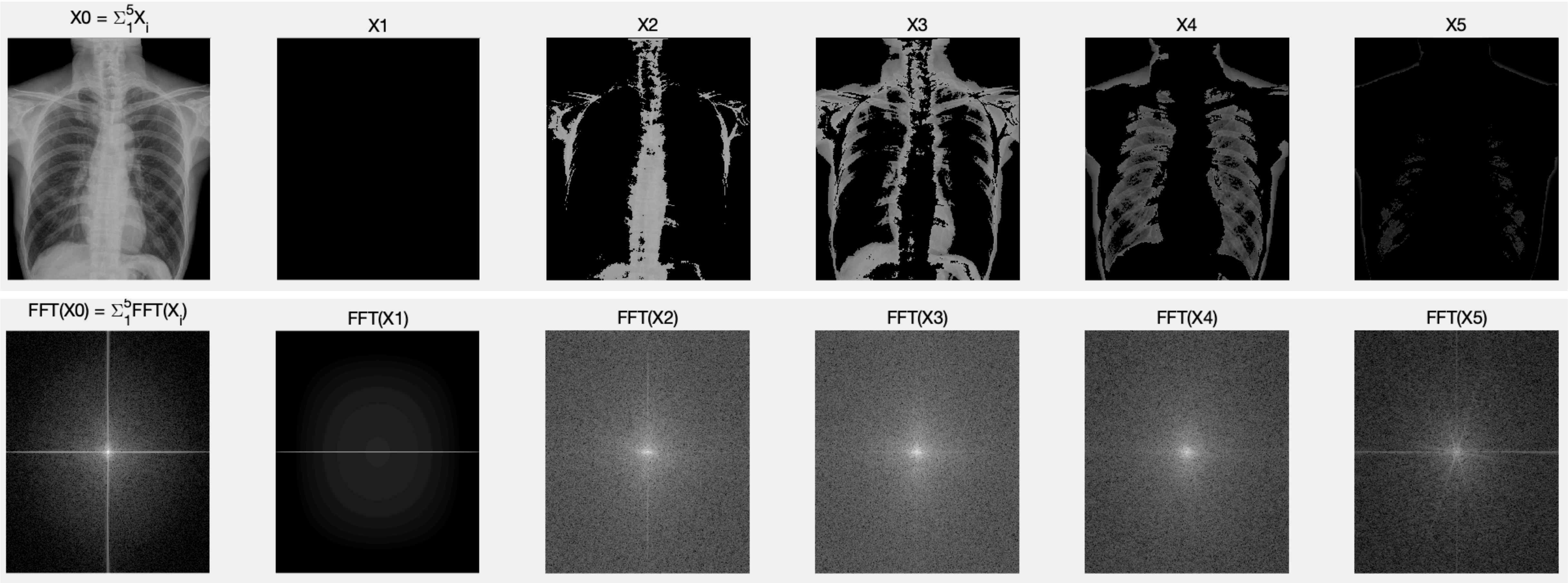
Using the polar coordinates, then:

$$x = r\cos\theta \qquad y = r\sin\theta \qquad u = \omega\cos\varphi \qquad v = \omega\sin\varphi$$

$$f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \varphi + \theta_0)$$

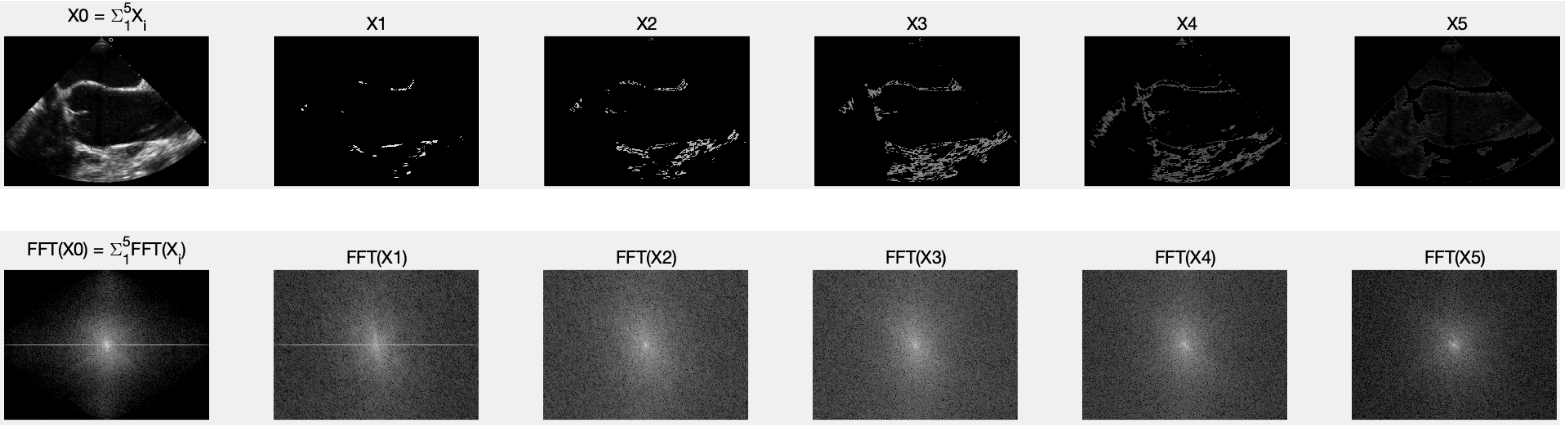
Properties of the 2-D DFT and IDFT

- Linearity



Properties of the 2-D DFT and IDFT

- Linearity



Properties of the 2-D DFT and IDFT

- Periodicity

As in the 1-D case, the 2-D Fourier transform and its inverse are infinitely periodic in the u and v directions; that is:

$$F(u, v) = F(u + k_1 M, v) = F(u, v + k_2 N) = F(u + k_1 M, v + k_2 N)$$

and

$$f(x, y) = f(x + k_1 M, y) = f(x, y + k_2 N) = f(x + k_1 M, y + k_2 N)$$

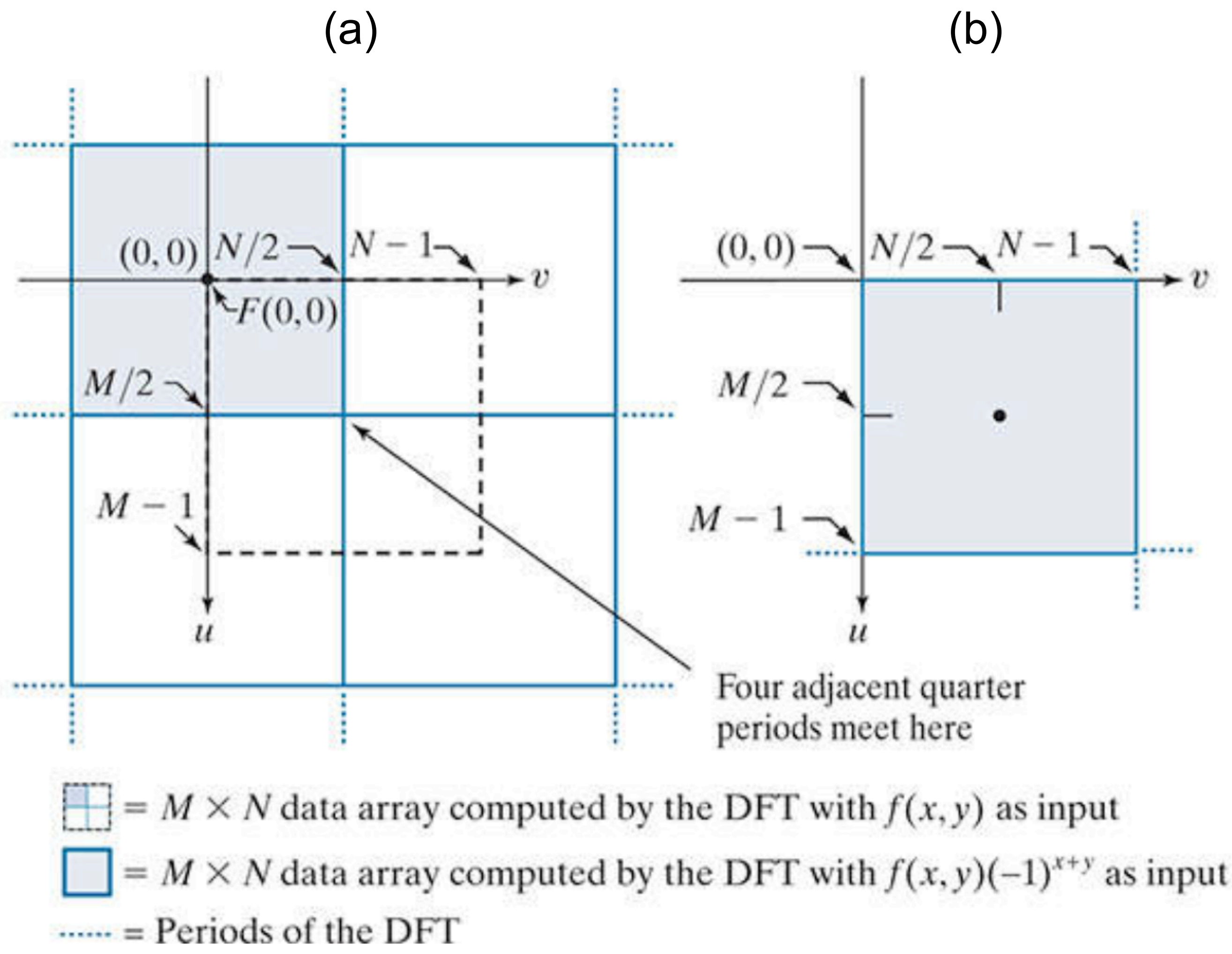
where k_1 and k_2 are integers.

Properties of the 2-D DFT and IDFT

For display, it is common practice to shift the original point to the center (M/2, N/2). According to the shift theorem, we need to multiply the input image function by $(-1)^{x+y}$ prior to computing the Fourier transform:

$$f(x, y)(-1)^{x+y} \Leftrightarrow F(u - M/2, v - N/2)$$

Properties of the 2-D DFT and IDFT



Centering the Fourier transform

(a) A 2-D DFT showing an infinite number of periods. The area within the dashed rectangle is the data array, $F(u, v)$, obtained with 2-D DFT with an image $f(x, y)$ as the input. This array consist of four quarter periods.

(b) Shifted array obtained by multiplying $f(x, y)$ by $(-1)^{x+y}$ before computing $F(u, v)$. The data now contains one complete, centered period.

Properties of the 2-D DFT and IDFT

- Symmetry Properties

$$w(x, y) = w_e(x, y) + w_o(x, y)$$

where the even and odd parts are defined as:

$$w_e(x, y) \triangleq \frac{w(x, y) + w(-x, -y)}{2} \quad \text{and} \quad w_o(x, y) \triangleq \frac{w(x, y) - w(-x, -y)}{2}$$

where

$$w_e(x, y) = w_e(-x, -y) \quad \text{and} \quad w_o(x, y) = -w_o(-x, -y)$$

	Spatial Domain†		Frequency Domain†
1)	*f*(*x*,*y*) real	⇔	$F^*(u,v) = F(-u,-v)$
2)	*f*(*x*,*y*) imaginary	⇔	$F^*(-u,-v) = -F(u,v)$
3)	*f*(*x*,*y*) real	⇔	*R*(*u*,*v*) even; *I*(*u*,*v*) odd
4)	*f*(*x*,*y*) imaginary	⇔	*R*(*u*,*v*) odd; *I*(*u*,*v*) even
5)	$f(-x,-y)$ rea	⇔	$F^*(u,v)$ compl e
6)	$f(-x,-y)$ compl e	⇔	$F(-u,-v)$ compl e
7)	$f^*(x,y)$ compl e	⇔	$F^*(-u,-v)$ compl e
8)	*f*(*x*,*y*) real and even	⇔	*F*(*u*,*v*) real and even
9)	*f*(*x*,*y*) real and odd	⇔	*F*(*u*,*v*) imaginary and odd
10)	*f*(*x*,*y*) imaginary and even	⇔	*F*(*u*,*v*) imaginary and even
11)	*f*(*x*,*y*) imaginary and odd	⇔	*F*(*u*,*v*) real and odd
12)	*f*(*x*,*y*) complex and even	⇔	*F*(*u*,*v*) complex and even
13)	*f*(*x*,*y*) complex and odd	⇔	*F*(*u*,*v*) complex and odd

Some symmetry properties of the 2-D DFT and its inverse. $R(u,v)$ and $I(u,v)$ are the real and imaginary parts of $F(u,v)$, respectively. Use of the word *complex* indicates that a function has nonzero real and imaginary parts.

Fourier Spectrum and Phase Angle

- Fourier spectrum

$$|F(u,v)| = \left[R^2(u,v) + I^2(u,v)\right]^{1/2}$$

- Phase angle

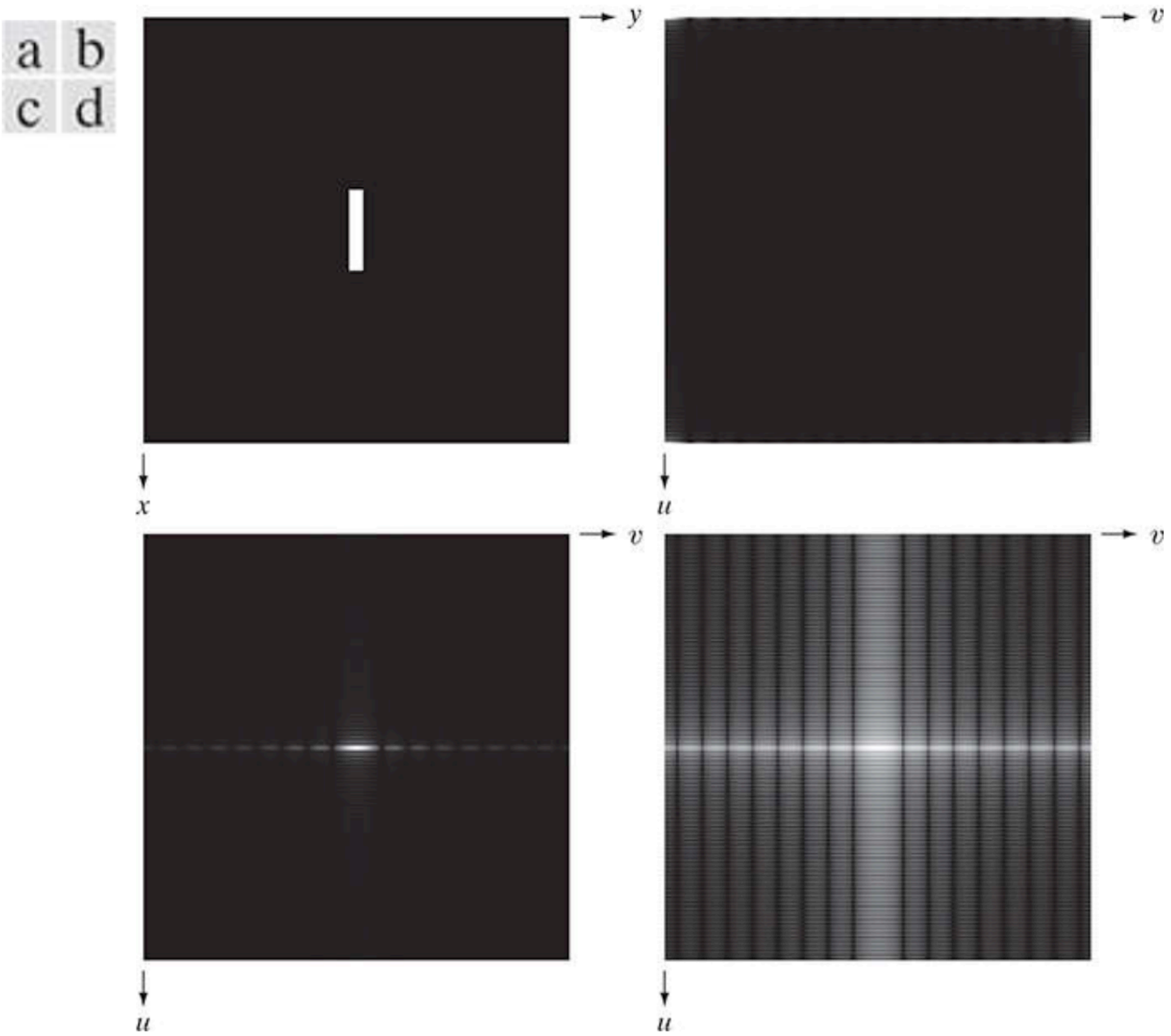
$$\phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$$

- Power spectrum

$$P(u,v) = |F(u,v)|^2 = R^2(u,v) + I^2(u,v)$$

where R(u, v) and I(u, v) are the real and imaginary parts of F(u, v), respectively

The spectrum of a rectangle

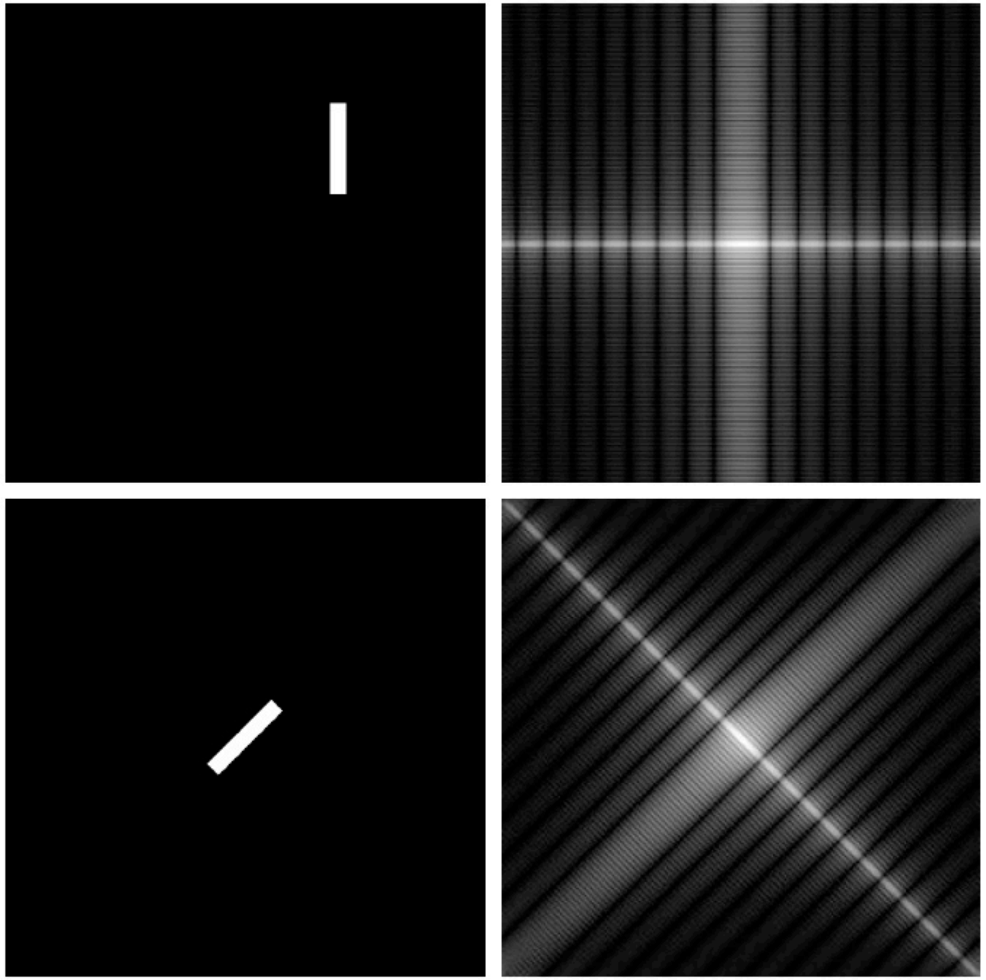


(a) Image.
(b) Spectrum, showing small, bright areas in the four corners (you have to look carefully to see them).
(c) Centered spectrum.
(d) Result after a log transformation.

The zero crossings of the spectrum are closer in the vertical direction because the rectangle in (a) is longer in that direction. The origin of the spatial and frequency domains at the top left

Translation and Rotation

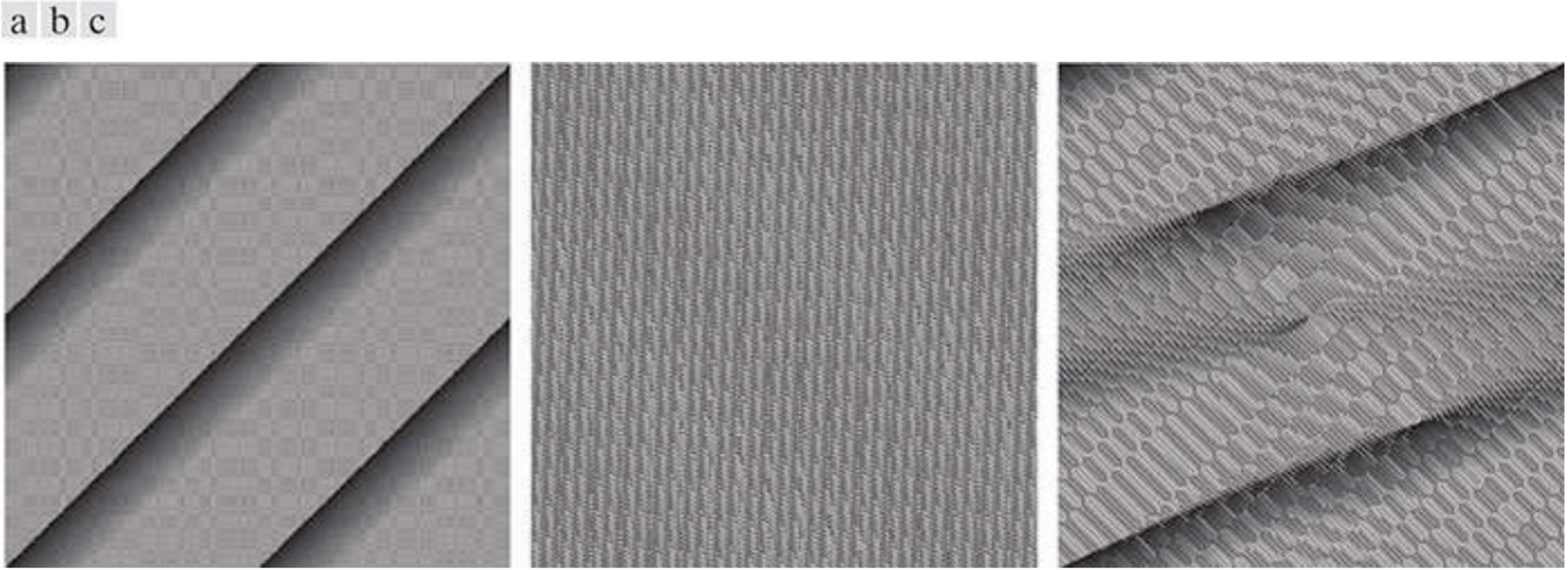
a b
c d



(a) The translated rectangle.
(b) Corresponding spectrum.
(c) Rotated rectangle.
(d) Corresponding spectrum.

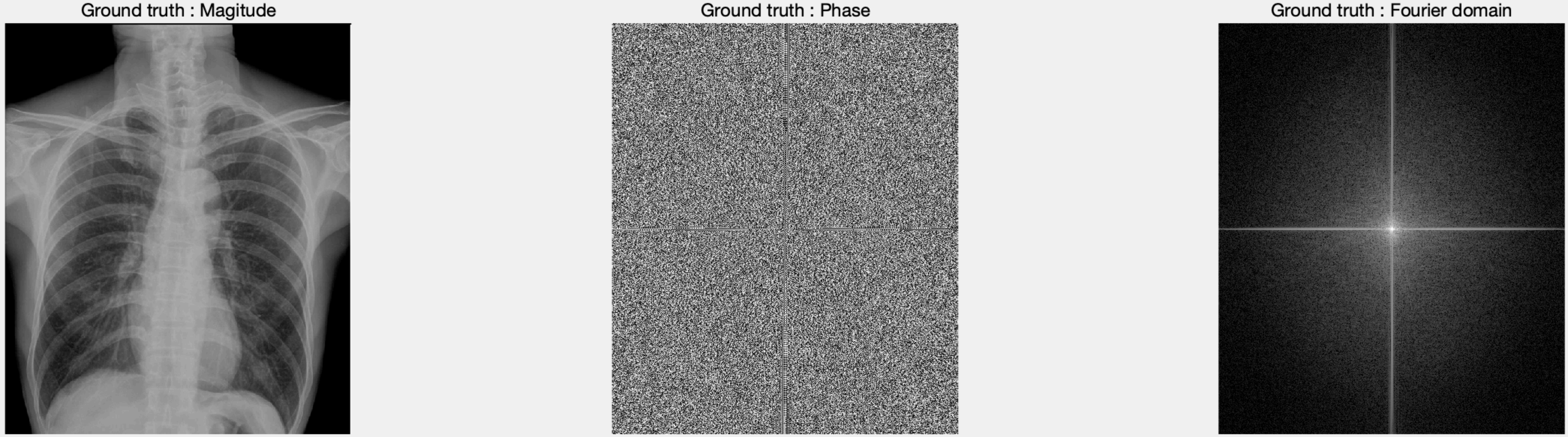
The spectrum of the translated rectangle is identical to the spectrum of the original image. The spectrum is insensitive to image translation (the absolute value of the exponential term is 1), but it rotates by the same angle of a rotated image. The components of the spectrum of the DFT determine the amplitudes of the sinusoids that combine to form an image.

Translation and Rotation



Phase angle images of (a) centered, (b) translated, and (c) rotated rectangles. The contribution of the phase components is less intuitive, it is just as important to spectrum.

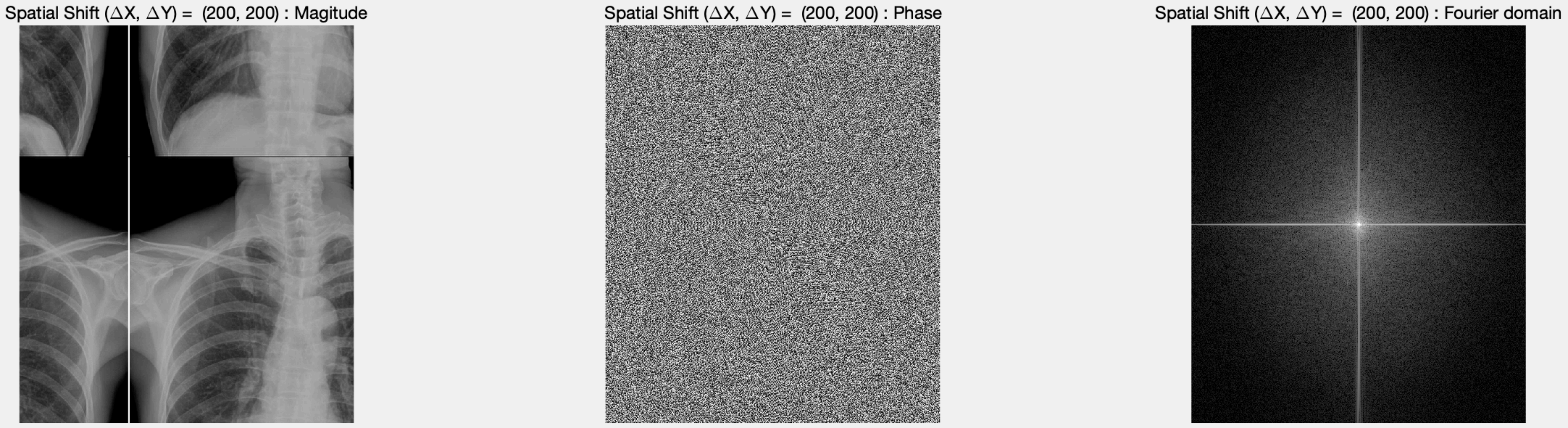
Amplitude and phase of X-ray images



(a) (b) (c)

(a) The original CT image. (b) The DFT phase of original CT image. (c) The Fourier domain of CT image.

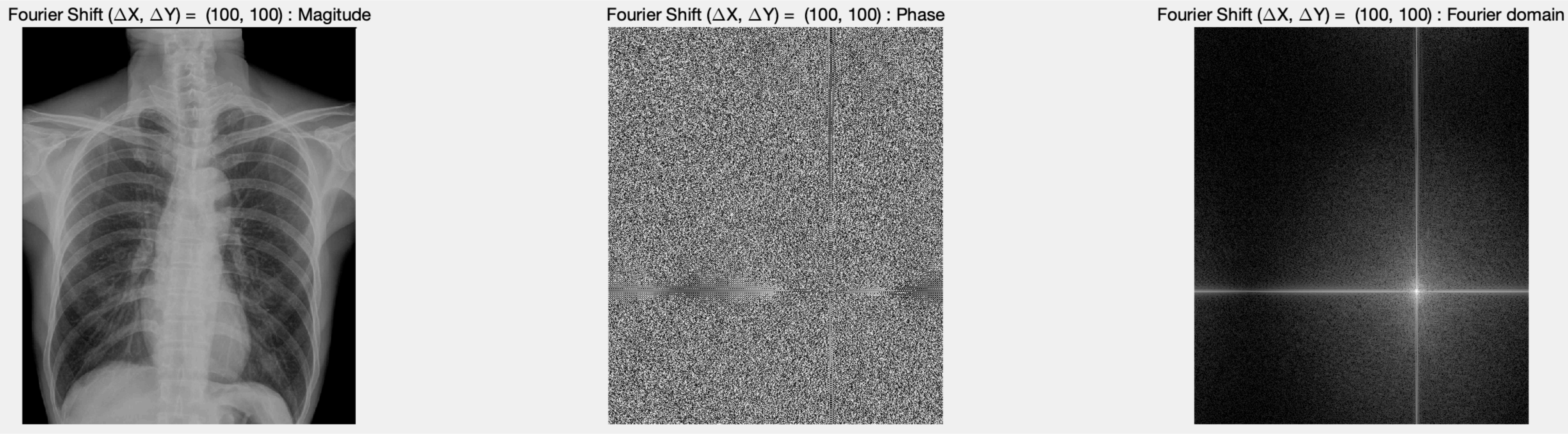
Spatial-shift in medical images



(a) (b) (c)

(a) CT image with spatial-shift. (b) The DFT phase of CT image with spatial shift. (c) The Fourier domain CT image with spatial shift.

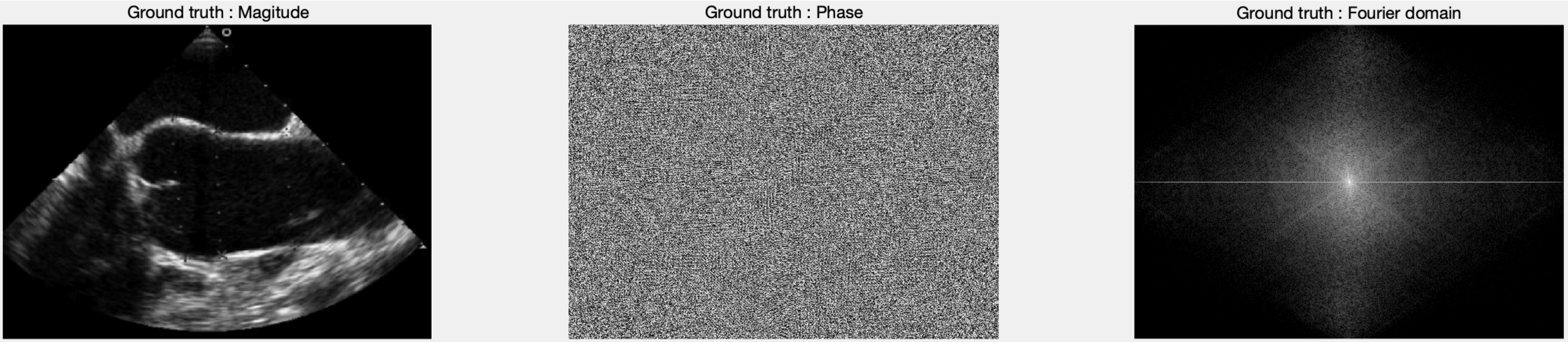
Frequency-shift in medical images



(a) (b) (c)

(a) The IDFT of (c). (b) The DFT phase of CT image with frequency shift. (c) The Fourier domain of CT image with frequency shift.

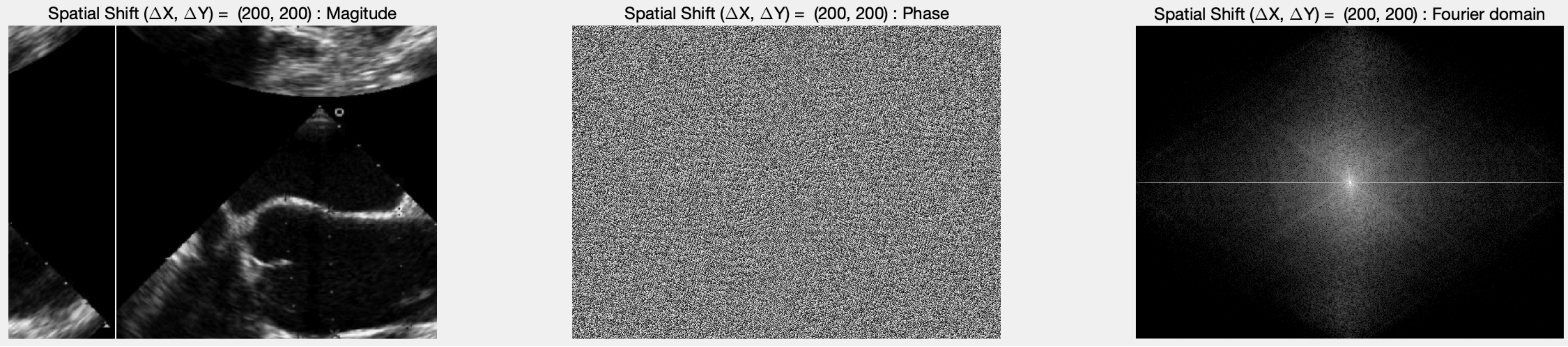
Amplitude and phase of ultrasound images



(a) (b) (c)

(a) The original ultrasound image. (b) The DFT phase of original ultrasound image. (c) The Fourier domain of ultrasound image.

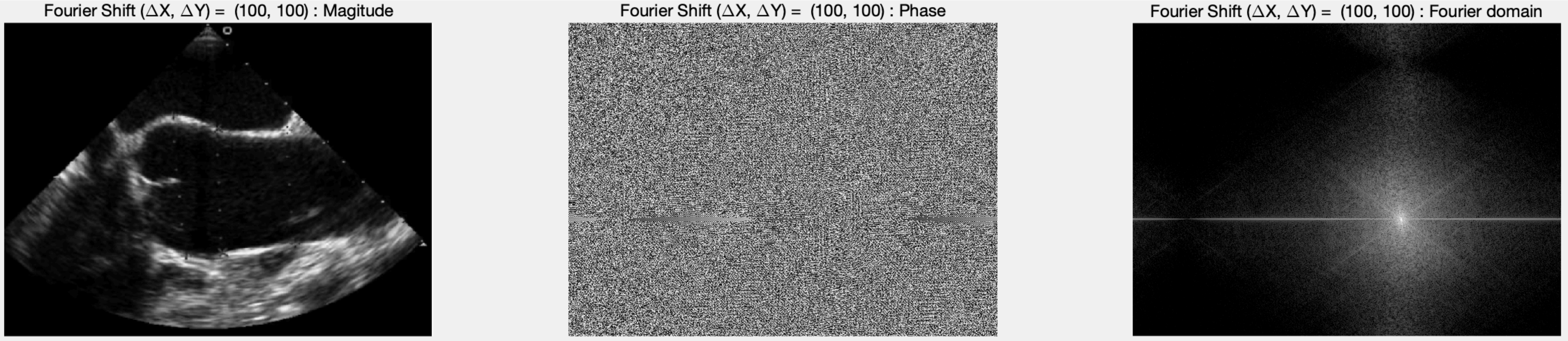
Spatial-shift in medical images



(a) (b) (c)

(a) Ultrasound image with spatial-shift. (b) The DFT phase of ultrasound image with spatial shift. (c) The Fourier domain ultrasound image with spatial shift.

Frequency-shift in medical images



(a) (b) (c)

(a) The IDFT of (c). (b) The DFT phase of ultrasound image with frequency shift. (c) The Fourier domain of ultrasound image with frequency shift.

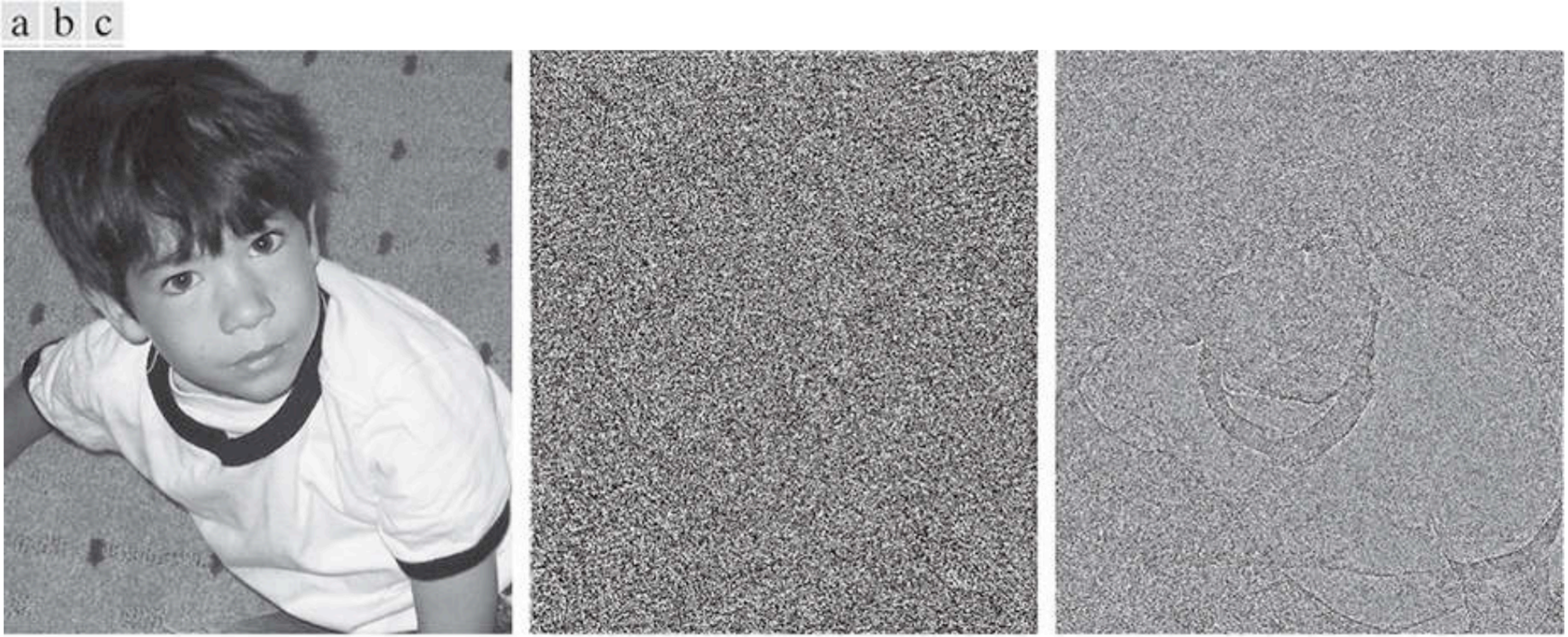
Contributions of the spectrum and phase angle to image formation

- The components of the spectrum of the DFT determine the amplitudes of the sinusoids that combine to form an image. At any given frequency in the DFT of an image, a large amplitude implies a greater prominence of a sinusoid of that frequency in the image.

- The phase is a measure of displacement of the various sinusoids with respect to their origin.

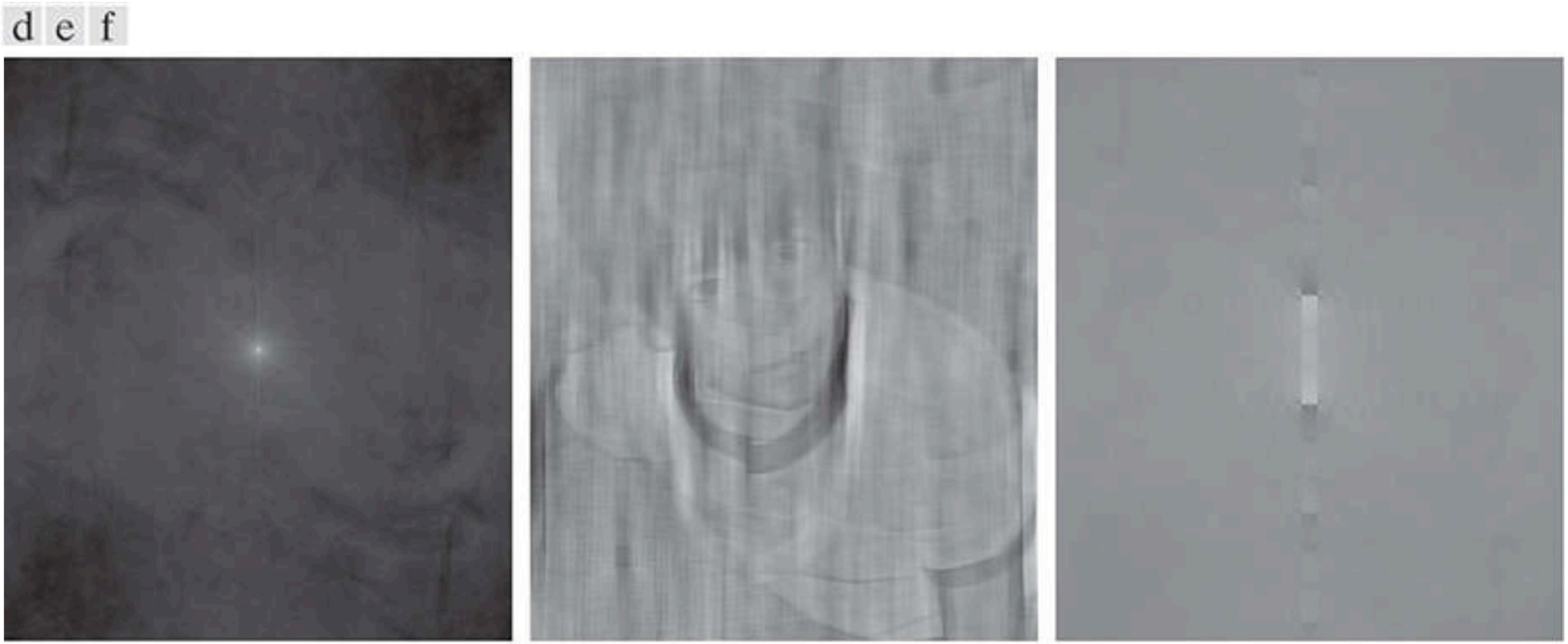
- The magnitude of the 2-D DFT is an array whose components determine the intensities in the image, and the corresponding phase is an array of angles that carry much of the information about where discernible objects are located in the image.

Contributions of the spectrum and phase angle to image formation



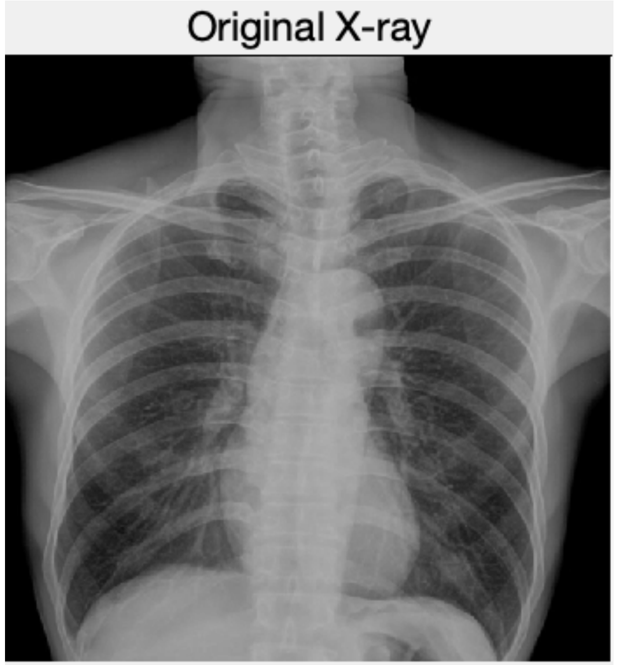
(a) Boy image. (b) Phase angle. (c) Boy image reconstructed using only its phase angle (all shape features are there, but the intensity information is missing because the spectrum was not used in the reconstruction).

Contributions of the spectrum and phase angle to image formation

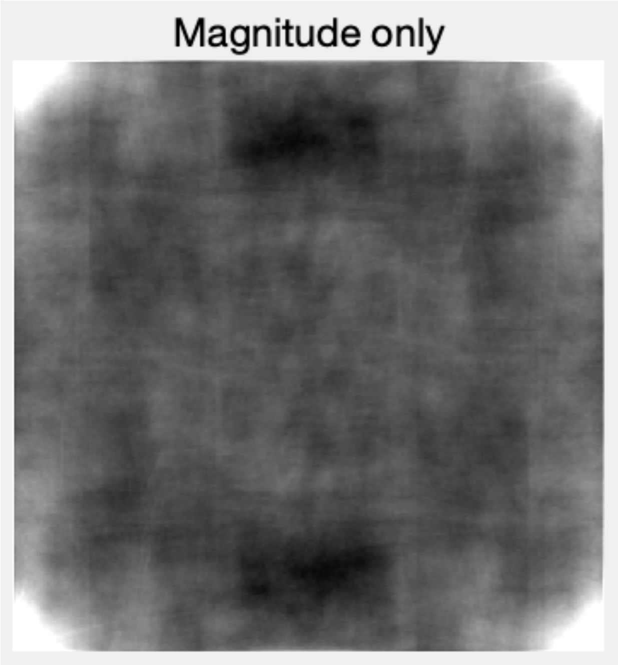


(d) Boy image reconstructed using only its spectrum. (e) Boy image reconstructed using its phase angle and the spectrum of the rectangle in Fig. 4.23(a) . (f) Rectangle image reconstructed using its phase and the spectrum of the boy's image.

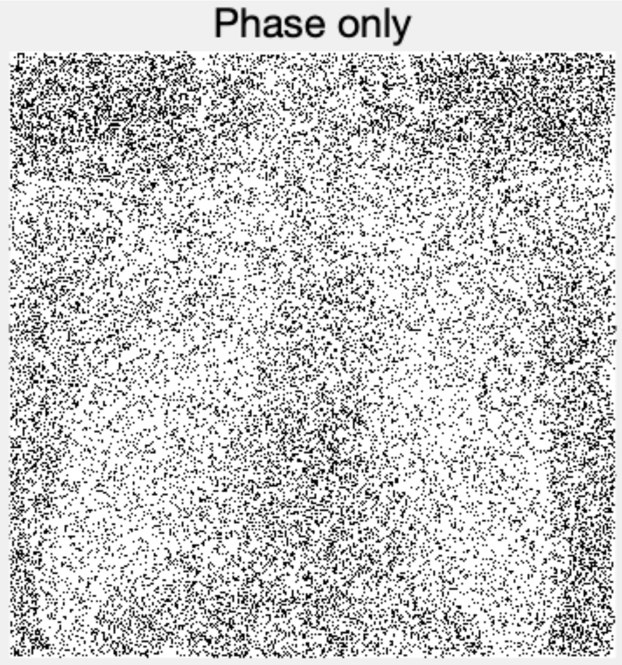
Contributions of the spectrum and phase angle to image formation



(a)



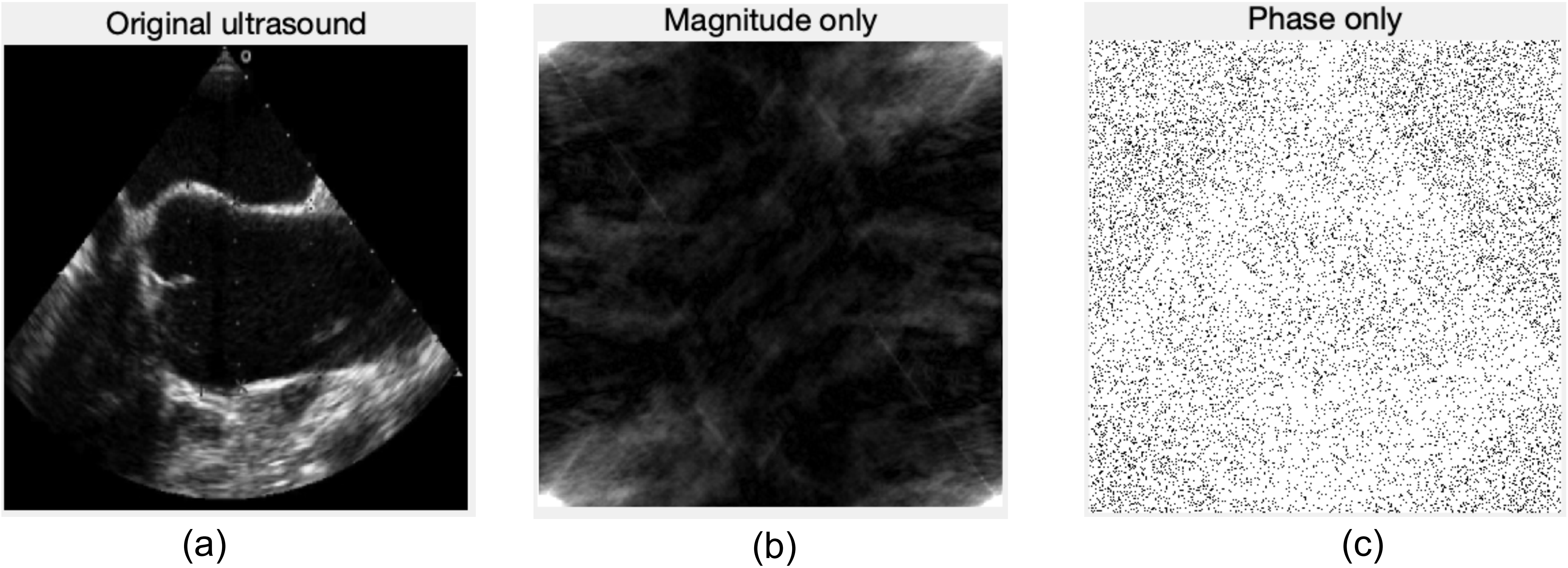
(b)



(c)

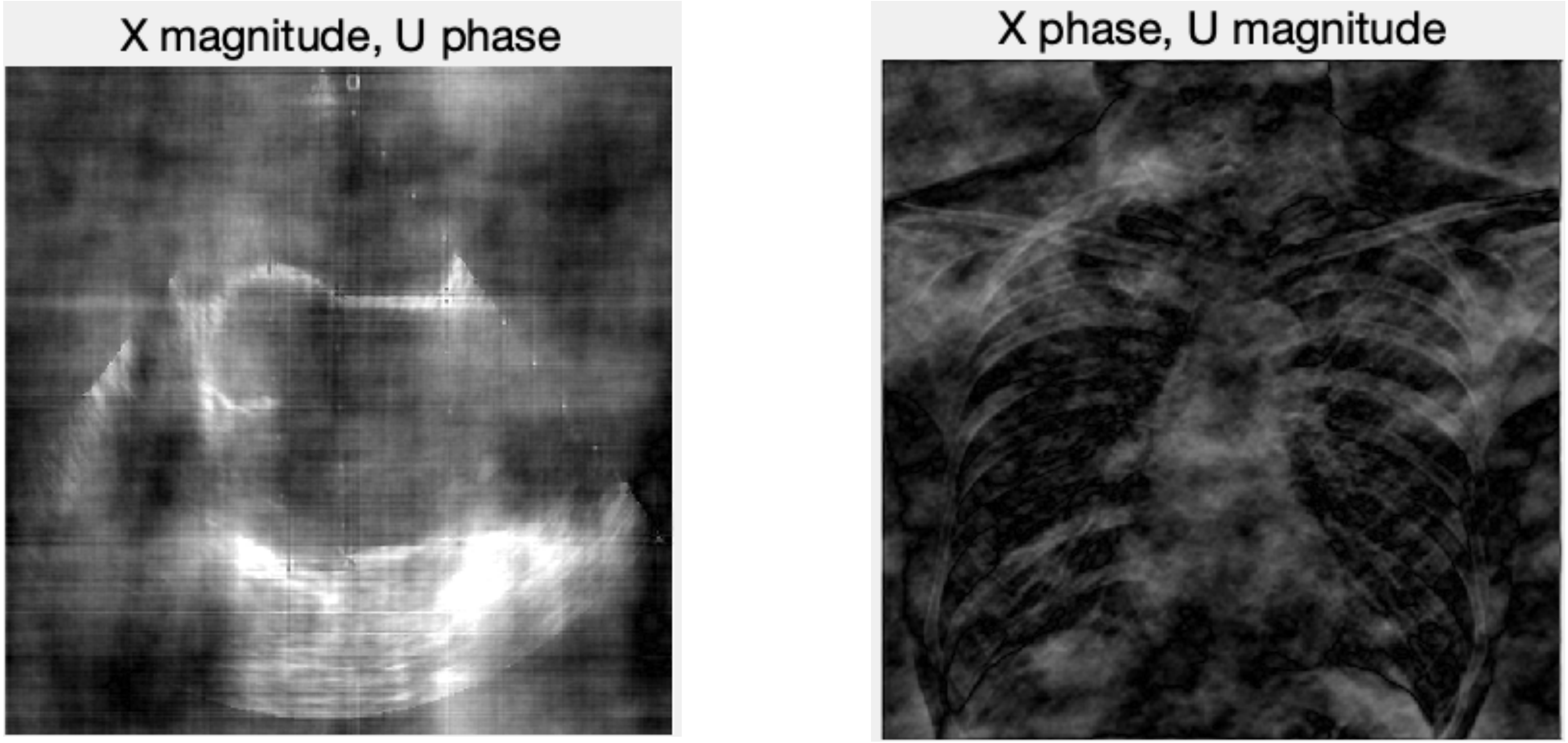
(a) The original X-ray image. (b) Reconstructing image with magnitude only. (c) Reconstructing image with phase only.

Contributions of the spectrum and phase angle to image formation



(a) The original ultrasound image. (b) Reconstructing image with magnitude only. (c) Reconstructing image with phase only.

Contributions of the spectrum and phase angle to image formation



(a) (b)

(a) Reconstructing image with X-ray image's magnitude and ultrasound image's phase. (b) Reconstructing image with X-ray image's phase and ultrasound image's magnitude.

The 2-D Convolution Theorem

- 2-D circular convolution

$$(f \star h)(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n)h(x-m, y-n)$$

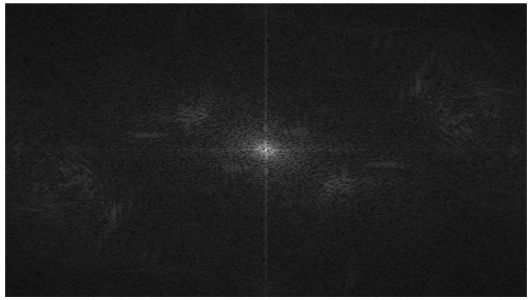
- The 2-D convolution theorem is given by

$$(f \star h)(x, y) \Leftrightarrow (F \cdot H)(u,v)$$

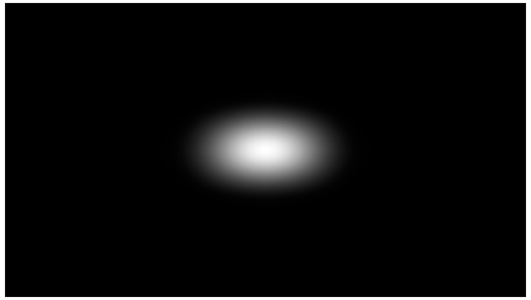
and, conversely,

$$(f \cdot h)(x, y) \Leftrightarrow \frac{1}{MN}(F \star H)(u,v)$$

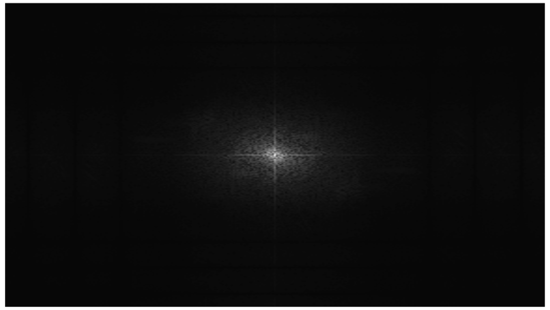
The 2-D Convolution Theorem

$f(x,y)$   $|F(s_x,s_y)|$

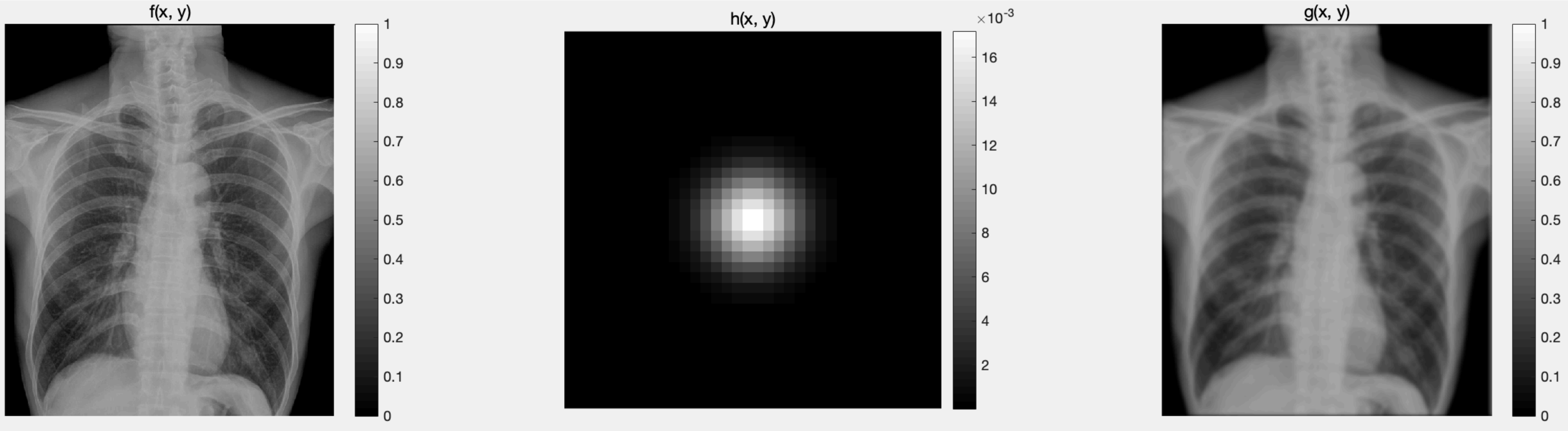
$*$ $\times$

$h(x,y)$   $|H(s_x,s_y)|$

$\Downarrow$ $\Downarrow$

$g(x,y)$   $|G(s_x,s_y)|$

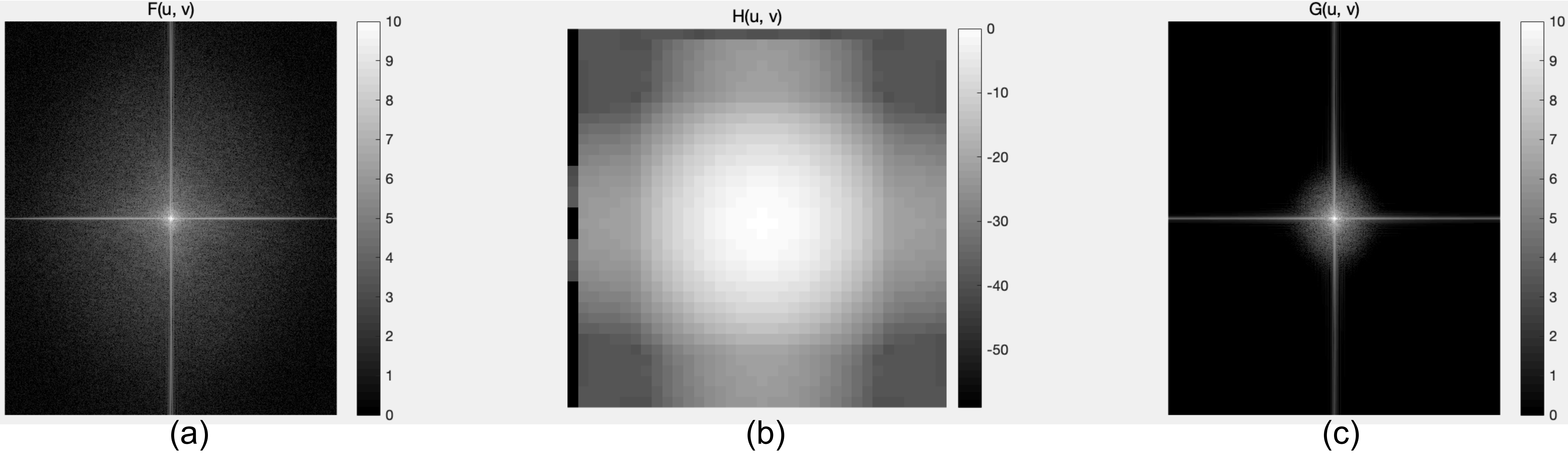
2-D Convolution Theorem



(a) (b) (c)

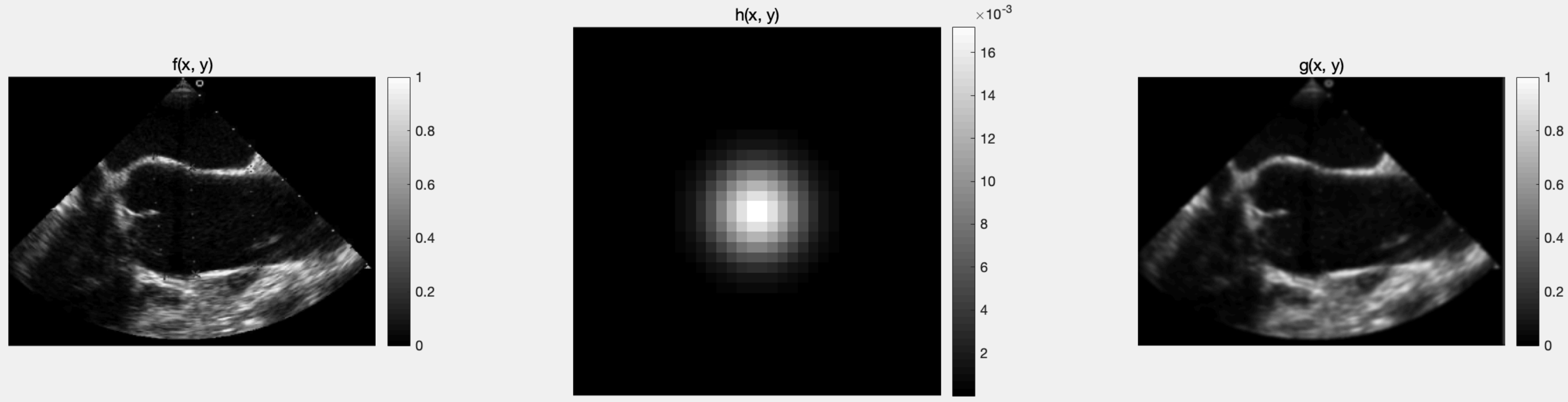
(a) Is the original image of X-ray image. (b) is the gaussian kernel. (c) is the convolution result between (a) and (b)

2-D Convolution Theorem



(a) is the DFT of original image. (b) is the DFT of the gaussian kernel. (c) is the DFT of resulted image. The y-axis is logarithmic magnitude of two-dimensional discrete Fourier transform

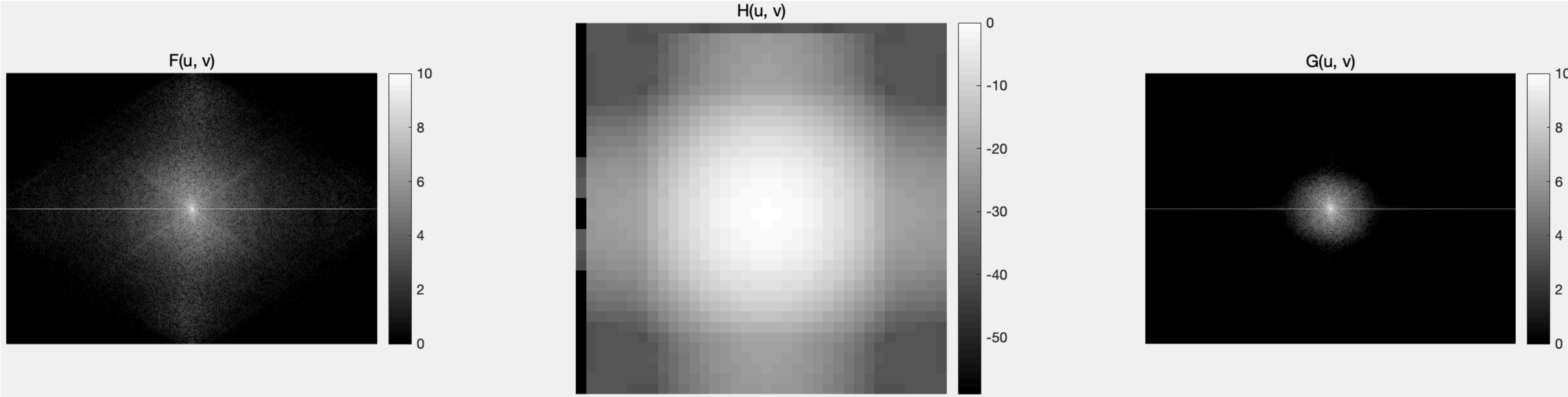
2-D Convolution Theorem



(a) (b) (c)

(a) Is the original image of ultrasound image. (b) is the gaussian kernel. (c) is the convolution result between (a) and (b)

2-D Convolution Theorem



(a) (b) (c)

(a) is the DFT of original image. (b) is the DFT of the gaussian kernel. (c) is the DFT of resulted image. The y-axis is logarithmic magnitude of two-dimensional discrete Fourier transform

Summary of 2-D DFT

	Name	Expression(s)
1)	Discrete Fourier transform (DFT) of *f*(*x*,*y*)	$F(u,v)=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$
2)	Inverse discrete Fourier transform (IDFT) of *F*(*u*,*v*)	$f(x,y)=\frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$
3)	Spectrum	$\lvert F(u,v)\rvert=\left[R^2(u,v)+I^2(u,v)\right]^{1/2} \qquad R=\mathrm{Real}(F);\ I=\mathrm{Imag}(F)$
4)	Phase angle	$\phi(u,v)=\tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$
5)	Polar representation	$F(u,v)=\lvert F(u,v)\rvert e^{j\phi(u,v)}$
6)	Power spectrum	$P(u,v)=\lvert F(u,v)\rvert^2$

Summary of 2-D DFT

7)	Average value	$\bar{f} = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y) = \frac{1}{MN}F(0,0)$
8)	Periodicity (k_1 and k_2 are integers)	$F(u,v) = F(u+k_1M,v) = F(u,v+k_2N)$ $= F(u+k_1,v+k_2N)$ $f(x,y) = f(x+k_1M,y) = f(x,y+k_2N)$ $= f(x+k_1M,y+k_2N)$
9)	Convolution	$(f \star h)(x,\ y) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)h(x-m,\ y-n)$
10)	Correlation	$(f \mathbin{☆} h)(x,\ y) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f^*(m,n)h(x+m,\ y+n)$
11)	Separability	The 2-D DFT can be computed by computing 1-D DFT transforms along the rows (columns) of the image, followed by 1-D transforms along the columns (rows) of the result. See **Section 4.11** .
12)	Obtaining the IDFT using a DFT algorithm	$MNf^*(x,\ y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F^*(u,v)e^{-j2\pi(ux/M+vy/N)}$

Summary of 2-D DFT pairs

	Name	DFT Pairs
1)	Symmetry properties	See **Table 4.1**
2)	Linearity	$af_1(x, y) + bf_2(x, y) \Leftrightarrow aF_1(u, v) + bF_2(u, v)$
3)	Translation (general)	$f(x, y)e^{j2\pi(u_0x/M + v_0y/N)} \Leftrightarrow F(u - u_0, v - v_0)$ $f(x - x_0, y - y_0) \Leftrightarrow F(u, v)e^{-j2\pi(ux_0/M + vy_0/N)}$
4)	Translation to center of the frequency rectangle, (*M*/2,*N*/2)	$f(x, y)(-1)^{x+y} \Leftrightarrow F(u - M/2, v - N/2)$ $f(x - M/2, y - N/2) \Leftrightarrow F(u, v)(-1)^{u+v}$
5)	Rotation	$f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \phi + \theta_0)$ $r = \sqrt{x^2 + y^2} \quad \theta = \tan^{-1}(y/x) \quad \omega = \sqrt{u^2 + v^2} \quad \phi = \tan^{-1}(v/u)$
6)	Convolution theorem†	$f \star h)(x, y) \Leftrightarrow (F \cdot H)(u, v)$ $(f \cdot h)(x, y) \Leftrightarrow (1/MN)[(F \star H)(u, v)]$

Summary of 2-D DFT pairs

7)	Correlation theorem†	$(f \star h)(x,y) \Leftrightarrow (F^* \cdot H)(u,v)$ $(f^* \cdot h)(x,y) \Leftrightarrow (1/MN)[(F \star H)(u,v)]$
8)	Discrete unit impulse	$\delta(x,y) \Leftrightarrow 1$ $1 \Leftrightarrow MN\delta(u,v)$
9)	Rectangle	$\mathrm{rec}[a,b] \Leftrightarrow ab \frac{\sin(\pi ua)}{(\pi ua)} \frac{\sin(\pi vb)}{(\pi vb)} e^{-j\pi(ua+vb)}$
10)	Sine	$\sin(2\pi u_0 x/M + 2\pi v_0 y/N) \Leftrightarrow \frac{jMN}{2}[\delta(u+u_0, v+v_0) - \delta(u-u_0, v-v_0)]$
11)	Cosine	$\cos(2\pi u_0 x/M + 2\pi v_0 y/N) \Leftrightarrow \frac{1}{2}[\delta(u+u_0, v+v_0) + \delta(u-u_0, v-v_0)]$
The following Fourier transform pairs are derivable only for continuous variables, denoted as before by t and z for spatial variables and by μ and v for frequency variables. These results can be used for DFT work by sampling the continuous forms.		
12)	Differentiation (the expressions on the right assume that $f(\pm\infty, \pm\infty) = 0$.	$\left(\frac{\partial}{\partial t}\right)^m \left(\frac{\partial}{\partial z}\right)^n f(t,z) \Leftrightarrow (j2\pi\mu)^m (j2\pi v)^n F(\mu,v)$ $\frac{\partial^m f(t,z)}{\partial t^m} \Leftrightarrow (j2\pi\mu)^m F(\mu,v); \frac{\partial^n f(t,z)}{\partial z^m} \Leftrightarrow (j2\pi v)^n F(\mu,v)$
13)	Gaussian	$A2\pi\sigma^2 e^{-2\pi^2\sigma^2(t^2+z^2)} \Leftrightarrow Ae^{-(\mu^2+v^2)/2\sigma^2}$ (A is a const a

Thanks!