上海交通大學
SHANGHAI JIAO TONG UNIVERSITY

Institute of Media, Information, and Network

生命科学技术学院
School of Life Sciences and Biotechnology

CT Reconstruction

Wenrui Dai
戴文睿
http://min.sjtu.edu.cn

计算机科学与工程系
上海交通大学

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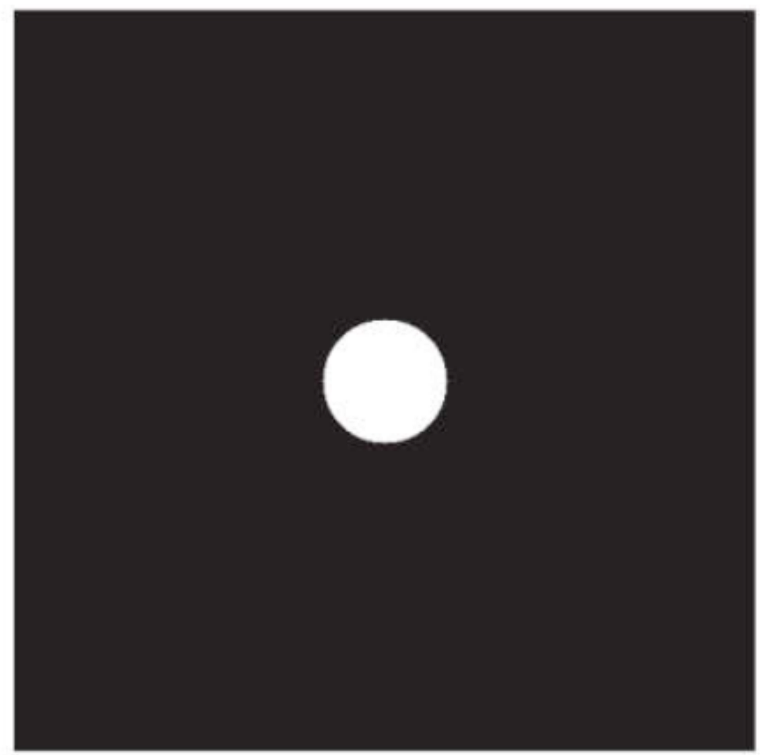
Principle of CT Reconstruction

- Let us begin with an image of flat region with a single object. Suppose that this image is a cross-section of a 3-D region of a human body. Assume also that the background in the image represents soft, uniform tissue, while the round object is a tumor, also uniform, but with higher X-ray absorption characteristics.



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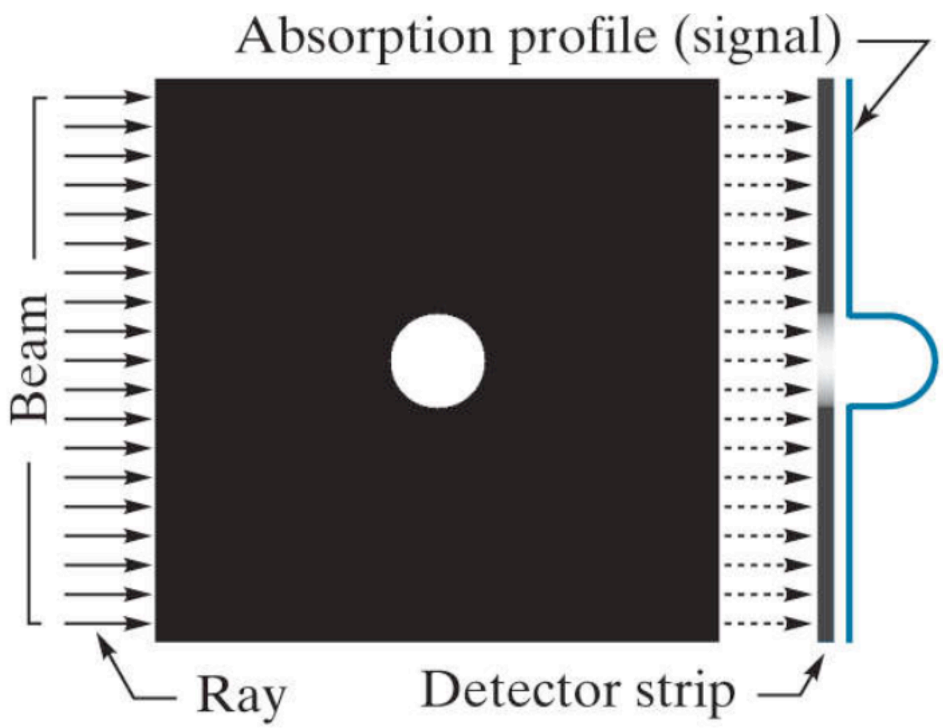
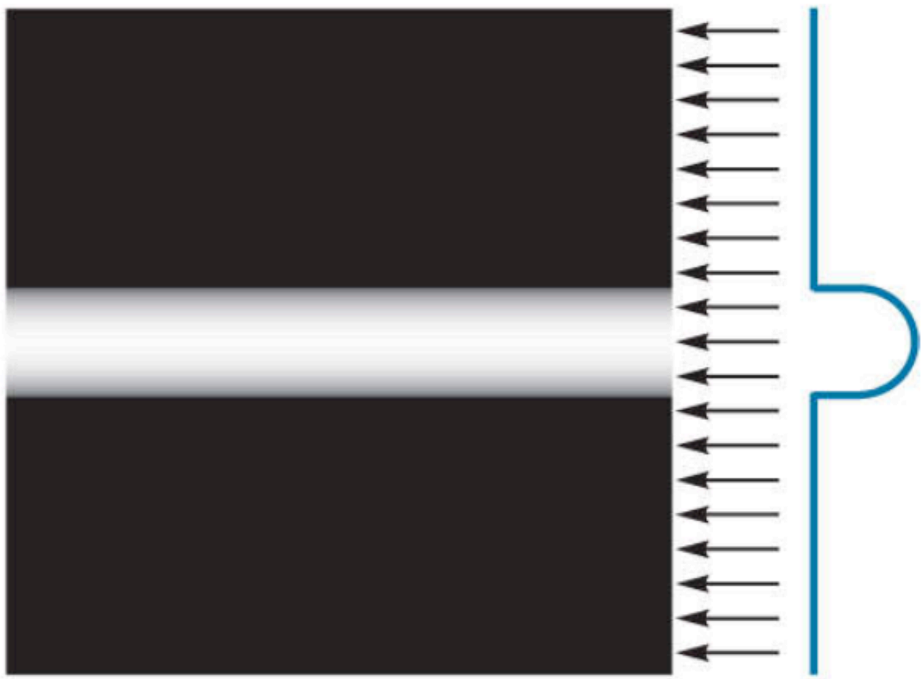
Principle of CT Reconstruction

- Suppose that we pass a thin, flat beam of X-rays from left to right (through the plane of the image), as the image show, and assume that the energy of the beam is absorbed more by the object than by the background.



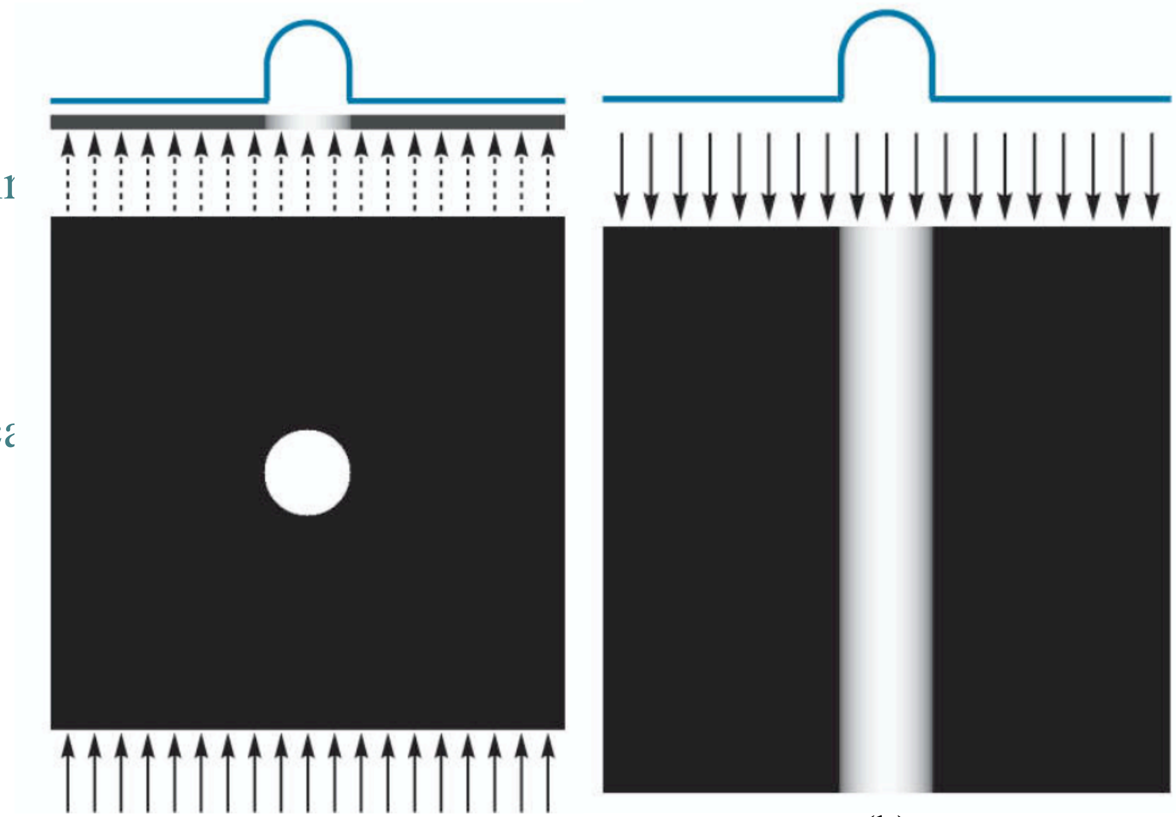
Principle of CT Reconstruction

- We have no way of determining from a single projection whether we are dealing with a single object, or a multitude of objects along the path of the beam.
- The approach is to project the 1-D signal back in the opposite direction from which the beam came, as the figure shows.



Principle of CT Reconstruction

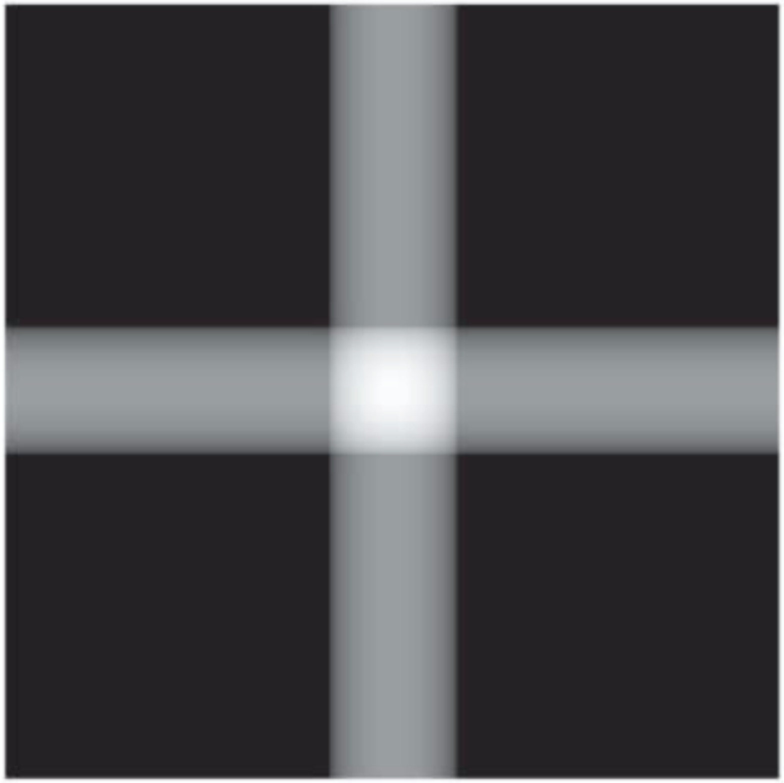
- Next, suppose that we rotate the position of the source-detector pair by 90°, as in figure (a). Repeating the procedure explained in the previous paragraph yields a backprojection image in the vertical direction, as shows in figure (b).



(a) (b)

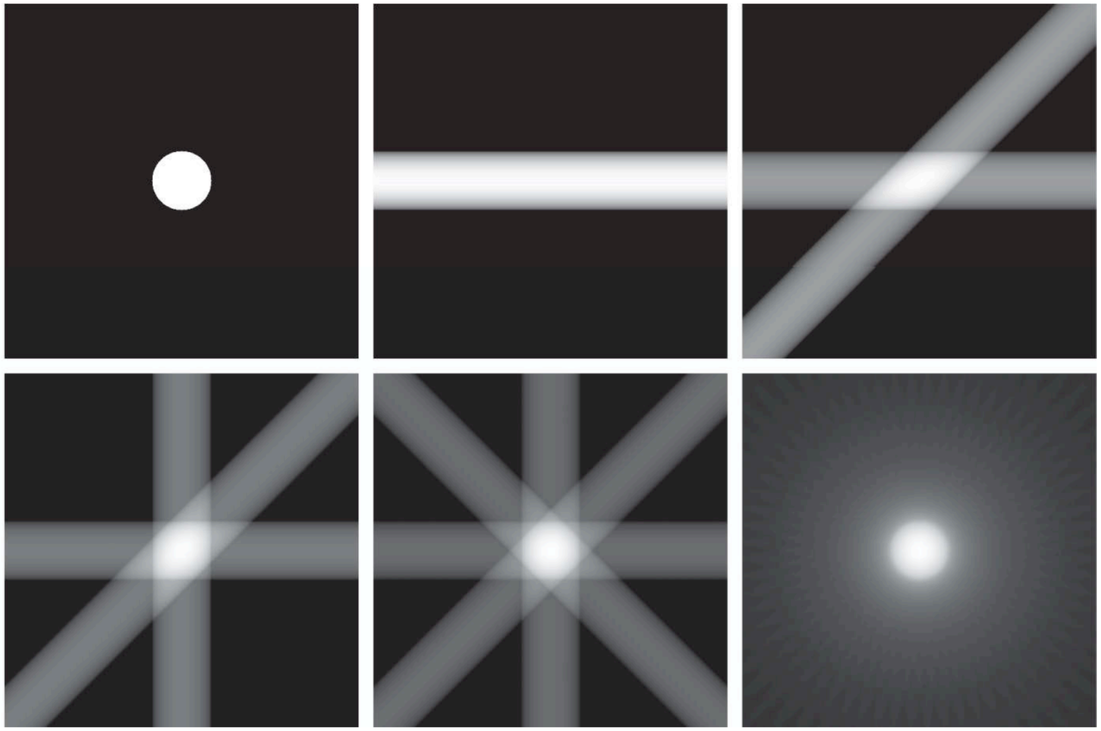
Principle of CT Reconstruction

- We continue the reconstruction by adding this result to the previous backprojection, resulting in figure right



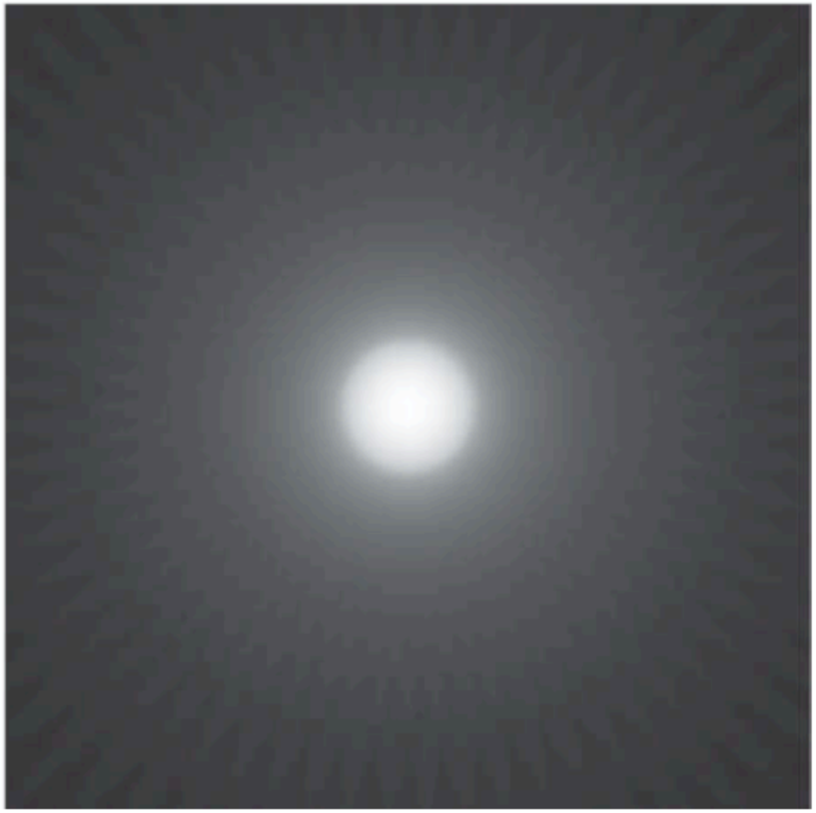
Principle of CT Reconstruction

- We should be able to learn more about the shape of the object in question by taking more views in the manner just described, as figure right shows.



Principle of CT Reconstruction

- Figure right was formed from 32 backprojections. Note, however, that while this reconstructed image is a reasonably good approximation to the shape of the original object, the image is blurred by a “halo” effect.



Principle of CT Reconstruction

- Backprojections of a planar region containing two objects

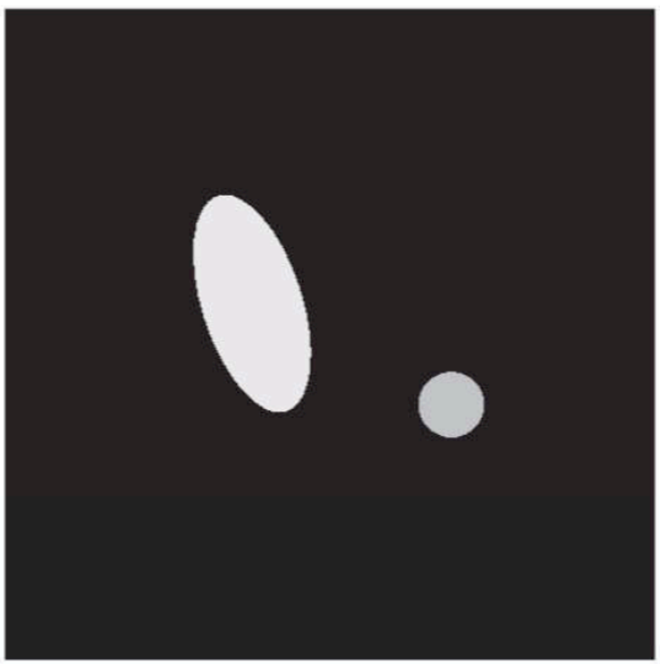
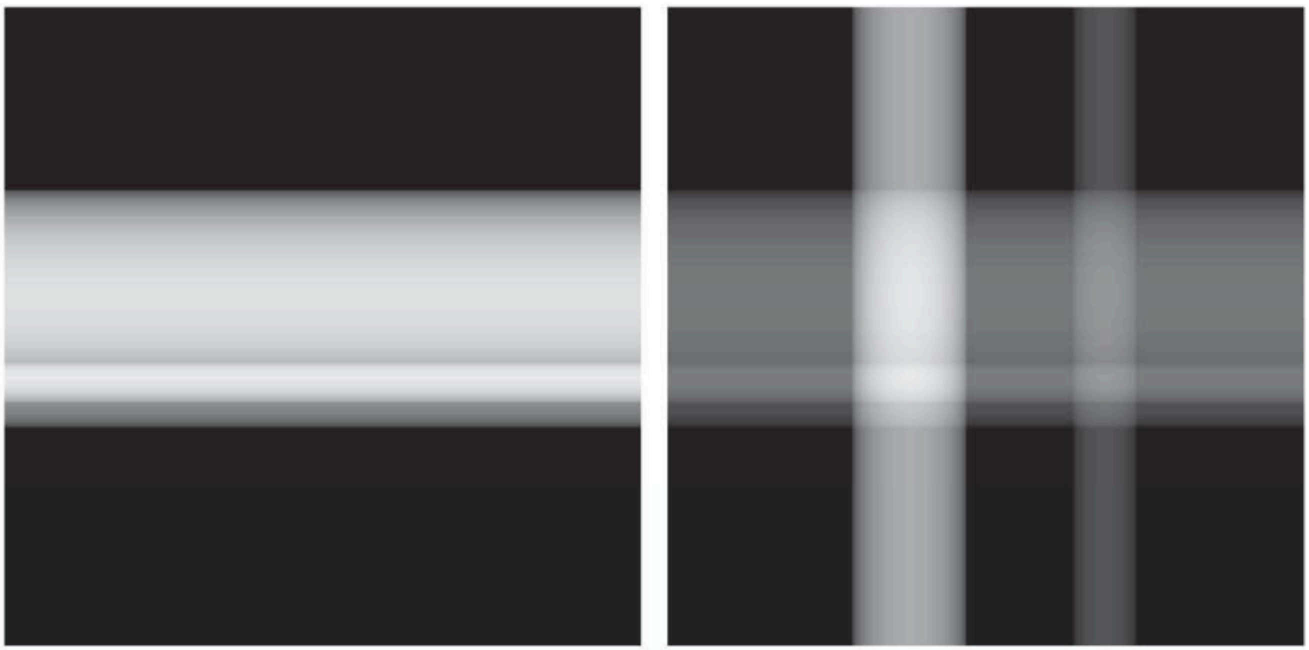


Figure left illustrates a region that contains two objects with different absorption properties (the larger object has higher absorption).

Principle of CT Reconstruction

- Backprojections of a planar region containing two objects



Reconstruction using 1, 2 backprojections respectively.

Principle of CT Reconstruction

- Backprojections of a planar region containing two objects

Reconstruction using 4, 32 and 64 backprojections respectively.

Principles of X-RAY Computed Tomography (CT)

- The basic mathematical concepts required for CT
  - The theoretical foundation of CT dates back to Johann Radon, a mathematician from Vienna who derived a method in 1917 for projecting a 2-D object along parallel rays, as part of his work on line integrals (the method now is referred to as the Radon transform, a topic we will discuss shortly).
  - Forty-five years later, Allan M. Cormack, a physicist at Tufts University, partially “rediscovered” these concepts and applied them to CT.

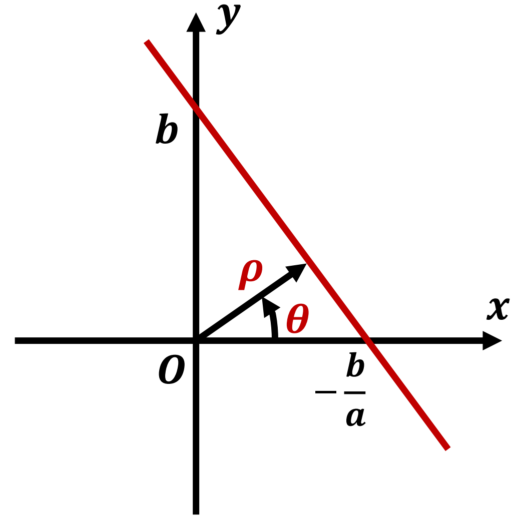
Principles of X-RAY Computed Tomography (CT)

- A straight line in Cartesian coordinates

$$y = ax + b$$

$$\Downarrow$$

$$x \cos\theta + y \sin\theta = \rho$$

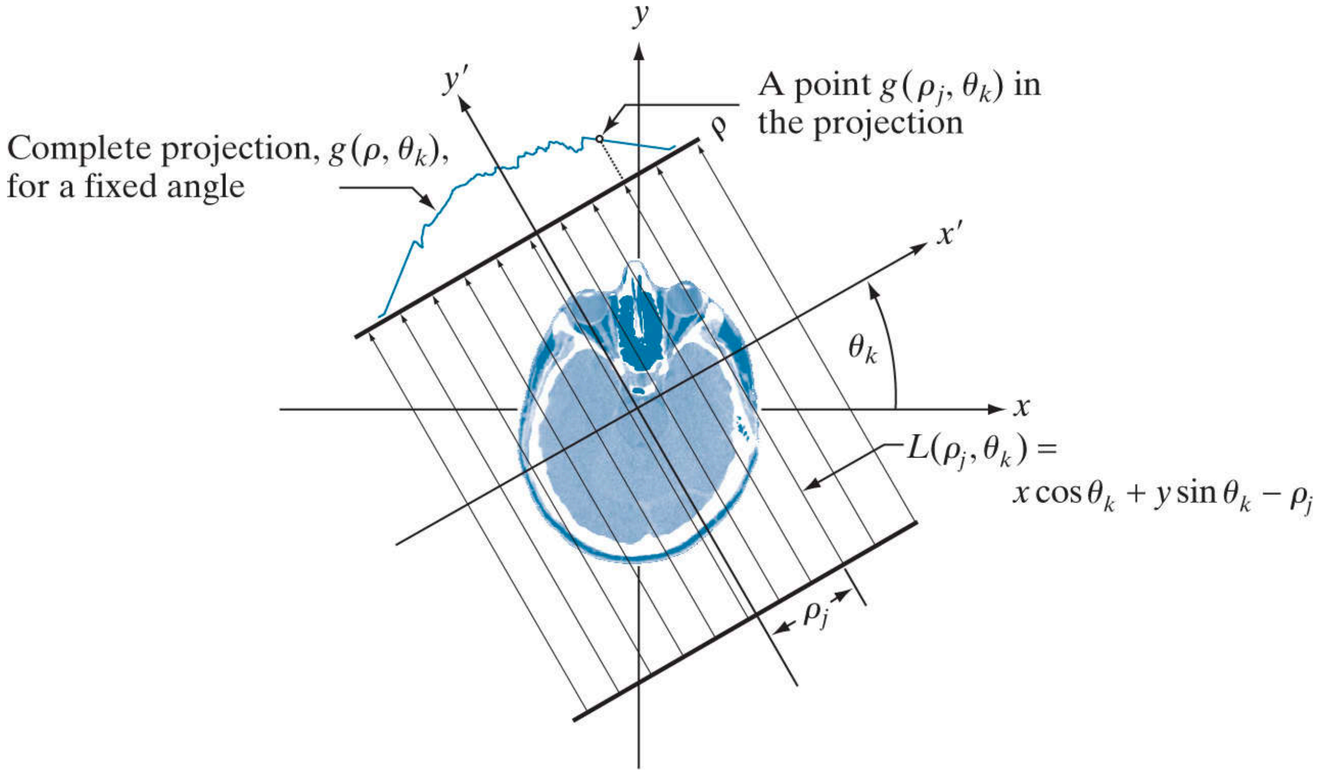


- Projection along the line

$$g(\rho_j, \theta_k) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y)\delta(x\cos\theta_k + y\sin\theta_k - \rho_j)\,dx\,dy$$

Principles of X-RAY Computed Tomography (CT)

- Geometry of parallel beam
- Projection along line with θ_k and ρ_j



R. C. Gonzalez and R. E. Woods, Digital Image Processing, Pearson, 2018.

Principles of X-RAY Computed Tomography (CT)

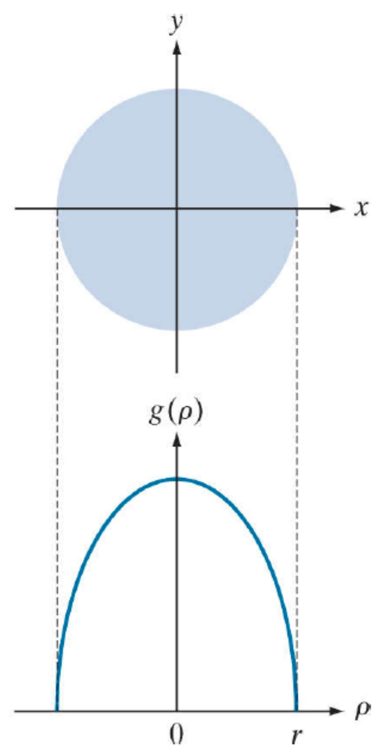
- Radon Transform

Projection of $f(x, y)$ along an arbitrary straight line in Cartesian coordinates

$$\mathcal{R}(f) = g(\rho, \theta) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y)\delta(x\cos\theta + y\sin\theta - \rho)\, dx\, dy$$

- Discrete variables (x, y)

$$\mathcal{R}(f) = g(\rho, \theta) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y)\delta(x\cos\theta + y\sin\theta - \rho)$$



R. C. Gonzalez and R. E. Woods, Digital Image Processing, Pearson, 2018.

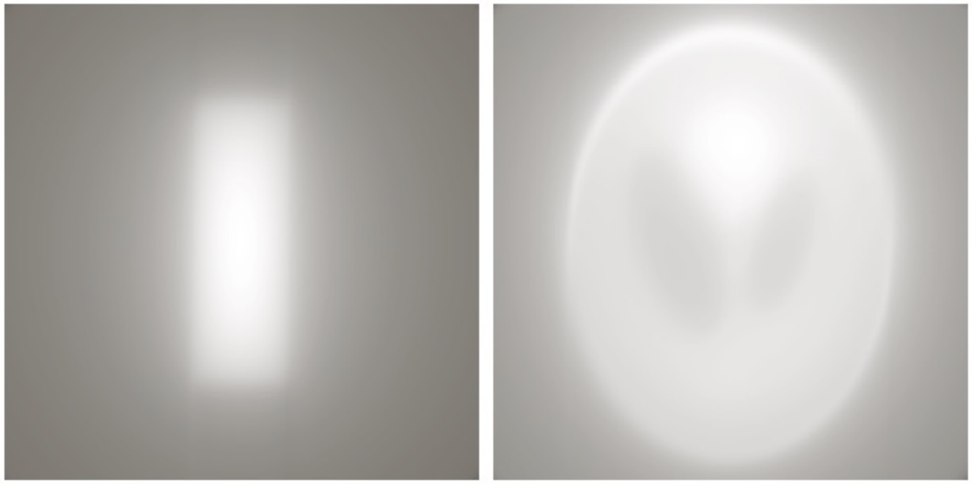
Backprojections

- Backprojected image from the Radon transform (sinogram)

$$f_{\theta_k}(x, y) = g(\rho, \theta_k) \qquad \rho = x\cos\theta_k + y\sin\theta_k$$

$$f(x, y) = \int_0^{\pi} f_\theta(x, y)\, d\theta$$

- Blurring: Halo effect



R. C. Gonzalez and R. E. Woods, Digital Image Processing, Pearson, 2018.

Backprojections

- The Fourier-Slice Theorem

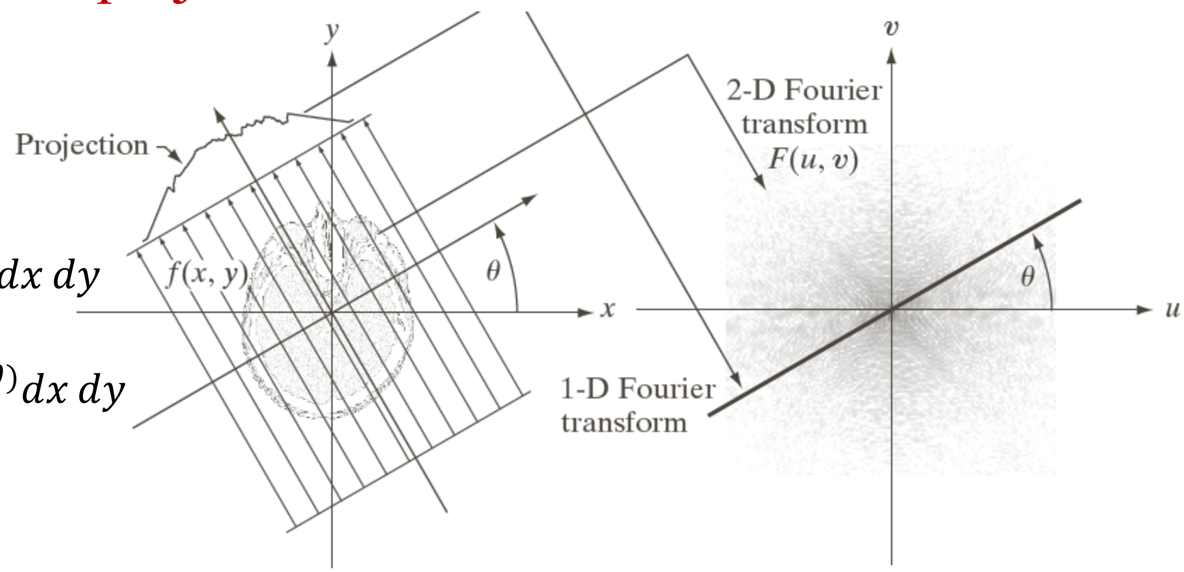
The 1-D Fourier transform of a projection is a slice of the 2-D Fourier transform.

$$G(\omega,\theta) = \int_{-\infty}^{+\infty} g(\rho,\theta) e^{-j2\pi\omega\rho} d\rho$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y) e^{-j2\pi\omega(x\cos\theta + y\sin\theta)} dx\, dy$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y) e^{-j2\pi(x\omega\cos\theta + y\omega\sin\theta)} dx\, dy$$

$$= F(\omega\cos\theta, \omega\sin\theta)$$



R. C. Gonzalez and R. E. Woods, Digital Image Processing, Pearson, 2018.

Filtered Backprojections

- Parallel-beam geometry

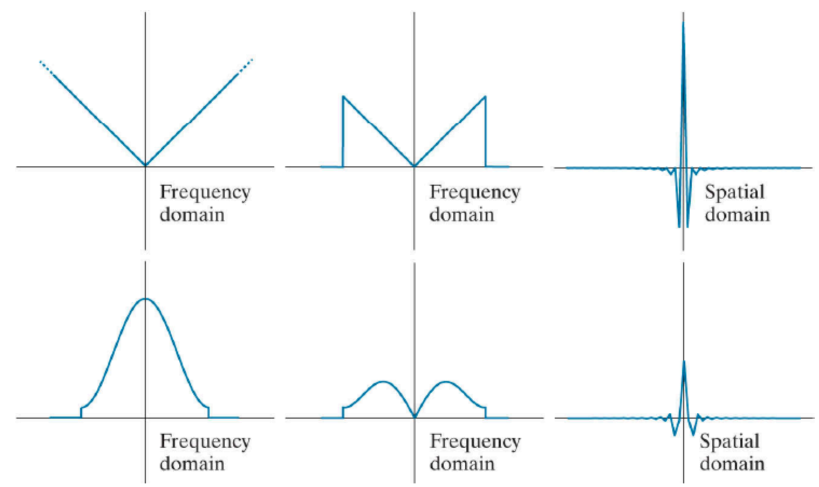
$$f(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(u,v)e^{j2\pi(ux+vy)}du\,dv$$

$$= \int_{0}^{2\pi}\int_{0}^{+\infty} G(\omega,\theta)e^{j2\pi(x\cos\theta+y\sin\theta)}\omega d\omega\,d\theta$$

$$= \int_{0}^{\pi}\int_{-\infty}^{+\infty} |\omega| G(\omega,\theta)e^{j2\pi\rho}d\omega\,d\theta$$

Ramp filter

Ramp filter — Band-limited ramp filter



Hamming window — Windowed ramp filter

R. C. Gonzalez and R. E. Woods, Digital Image Processing, Pearson, 2018.

Filtered Backprojections

- Fan-beam geometry

$$\boldsymbol{\theta} = \boldsymbol{\beta} + \boldsymbol{\alpha} \qquad \boldsymbol{\rho} = \boldsymbol{D}\sin\boldsymbol{\alpha}$$

- Backprojection formula

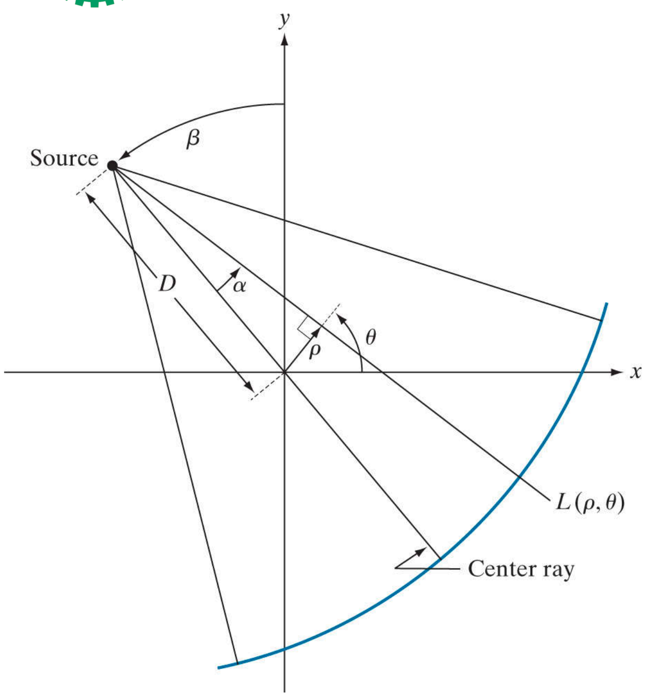
$$f(x,y) = \frac{1}{2}\int_0^{2\pi}\int_{-T}^{T} g(\rho,\theta)\, s(x\cos\theta + y\sin\theta - \rho)\, d\rho\, d\theta$$

$g(\rho,\theta) = 0$ for $|\rho| > T$

$s(\rho)$: the inverse Fourier transform of $|\omega|$

$$= \frac{1}{2}\int_0^{2\pi}\int_{-T}^{T} g(\rho,\theta)\, s(r\cos(\theta-\phi) - \rho)\, d\rho\, d\theta$$

Conversion to polar coordinates (r,ϕ)



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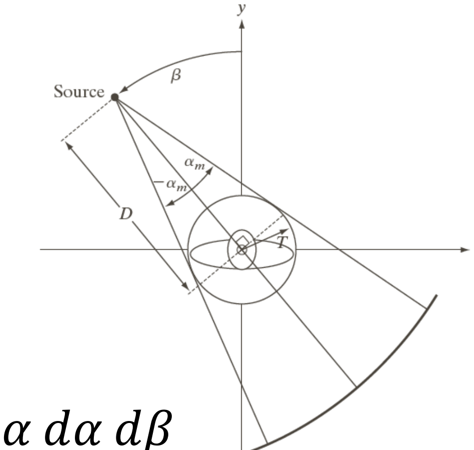
Filtered Backprojections



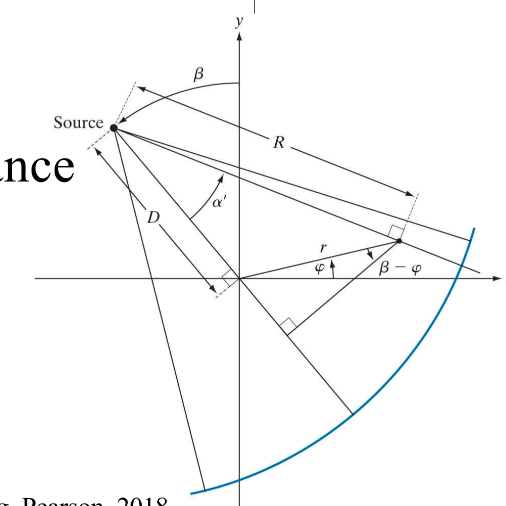
- Relate to α and β

$\alpha_m = \sin^{-1}(T/D)$

$$f(r,\phi) = \frac{1}{2}\int_0^{2\pi}\int_{-\alpha_m}^{\alpha_m} g(D\sin\alpha\,,\alpha+\beta)\,s(r\cos(\beta+\alpha-\phi)-D\sin\alpha)D\cos\alpha\,d\alpha\,d\beta$$

$p(\alpha,\beta)$

- Maximum value of α needed to encompass a region of interest
- Introduce the angle α' between the ray and center ray and the distance R from the source to an arbitrary point



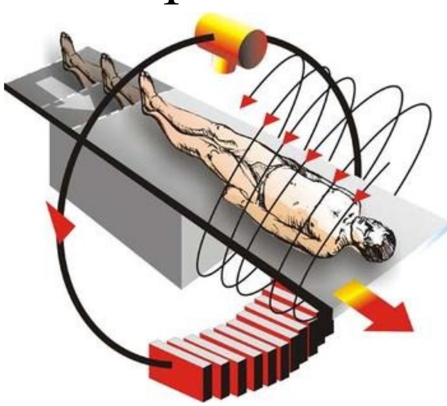
$$f(r,\phi) = \frac{1}{2}\int_0^{2\pi}\int_{-\alpha_m}^{\alpha_m} p(\alpha,\beta)s(R\sin(\alpha'-\alpha))D\cos\alpha\,d\alpha\,d\beta$$

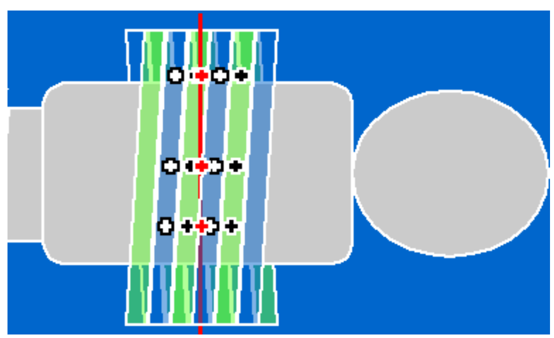
$$r\cos(\beta+\alpha-\phi) - D\sin\alpha = R\sin(\alpha'-\alpha)$$

R. C. Gonzalez and R. E. Woods, Digital Image Processing, Pearson, 2018.

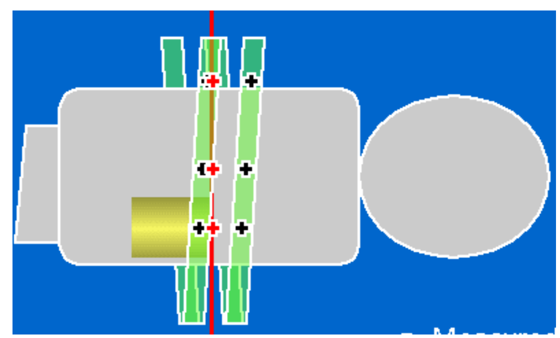
Helical (Spiral) CT Reconstruction

- The patient advances at a constant rate through the scanner gantry while the X-ray tube rotates continuously around the patient, tracing a spiral path through the patient.
- Interpolate to achieve reconstruction





180-degree linear interpolation

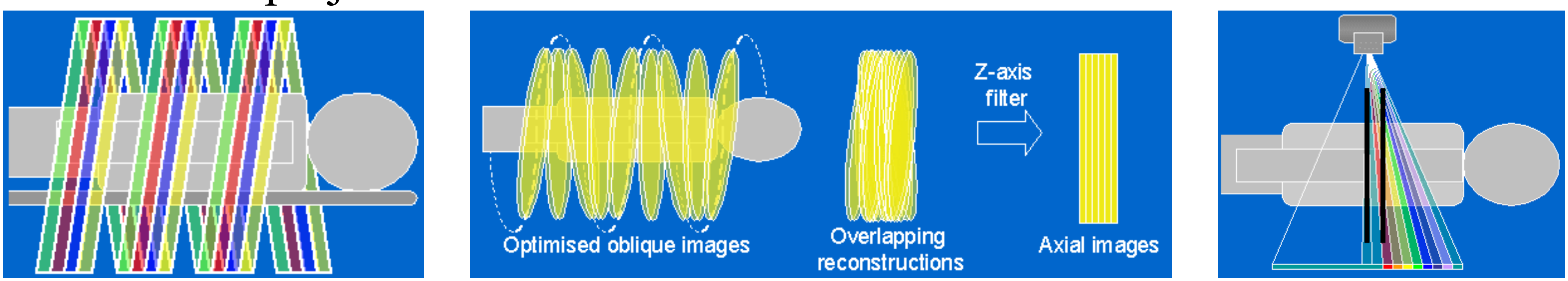


360-degree linear interpolation

C. N. Nordan, “Helical CT & Lung Cancer,” 2011. [Online] Available: http://www.personal.psu.edu/afr3/blogs/SIOW/2011/12/helical-ct-lung-cancer.html
N. Keat, “Helical and multi-sclie principles,” 2005. [Online] Available: http://www.impactscan.org/slides/impactcourse/helical_and_multi-slice_principles/index.html

Multi-Slice CT Reconstruction

- Linear interpolation:
  Interpolate projections from the same scan angle
- Z-filtering
  Image backprojected along oblique planes for Z-axis filtering
- 3-D backprojections



N. Keat, “Helical and multi-sclie principles,” 2005. [Online] Available: http://www.impactscan.org/slides/impactcourse/helical_and_multi-slice_principles/index.html

Low-Dose CT Reconstruction

- CT reconstruction from incomplete and noisy data
- TV regularization $J(f) = \int \|\nabla f\|_1 dx$

$$\min_f \frac{\lambda}{2}\|Pf - y\|_2^2 + J(f) \quad \text{s.t. } f_i \geq 0, \forall i$$

- Sparse representation

$$\min_{f,\alpha_i,D} (Pf - y)^T \Sigma^{-1}(Pf - y) + \beta\left(\sum_i \|\mathcal{E}_i f - D\alpha_i\|_2^2 + \lambda_i\|\alpha_i\|_0\right)$$

- Deep learning
- Relation to compressed sensing

Z. Tian, X. Jia, K. Yuan, T. Pan and S. B. Jiang, “Low dose CT reconstruction via edge-preserving total variation regularization,” Physics in Medicine & Biology, 2011, 56(18): 5949-5967
Q. Xu, X. Mou, L. Zhang, J. Hsieh, and G. Wang, “Low-dose X-ray CT reconstruction via dictionary learning,” IEEE Transactions on Medical Imaging, 2012, 31(9): 1682-1697.

Thank You!