SHANGHAI JIAO TONG UNIVERSITY

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生命科学技术学院

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小波与多分辨率处理

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Topical today

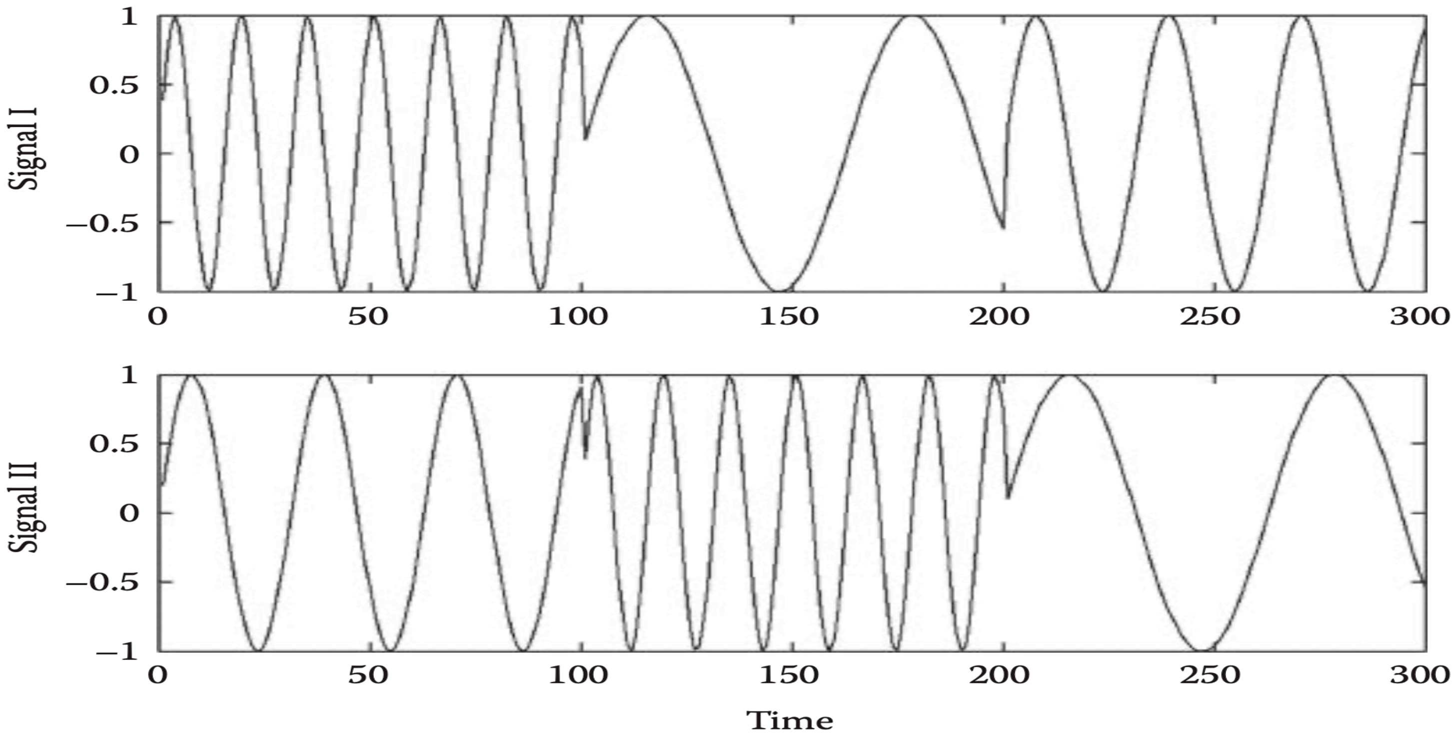
- A tour of wavelets
- Basic of multiresolution analysis
- Glance at multiscale geometry analysis and sparse representation

A Tour of Wavelets

From FT to STFT

- Concern over the use of the FT
  - Incapable of locating singularity

Example:

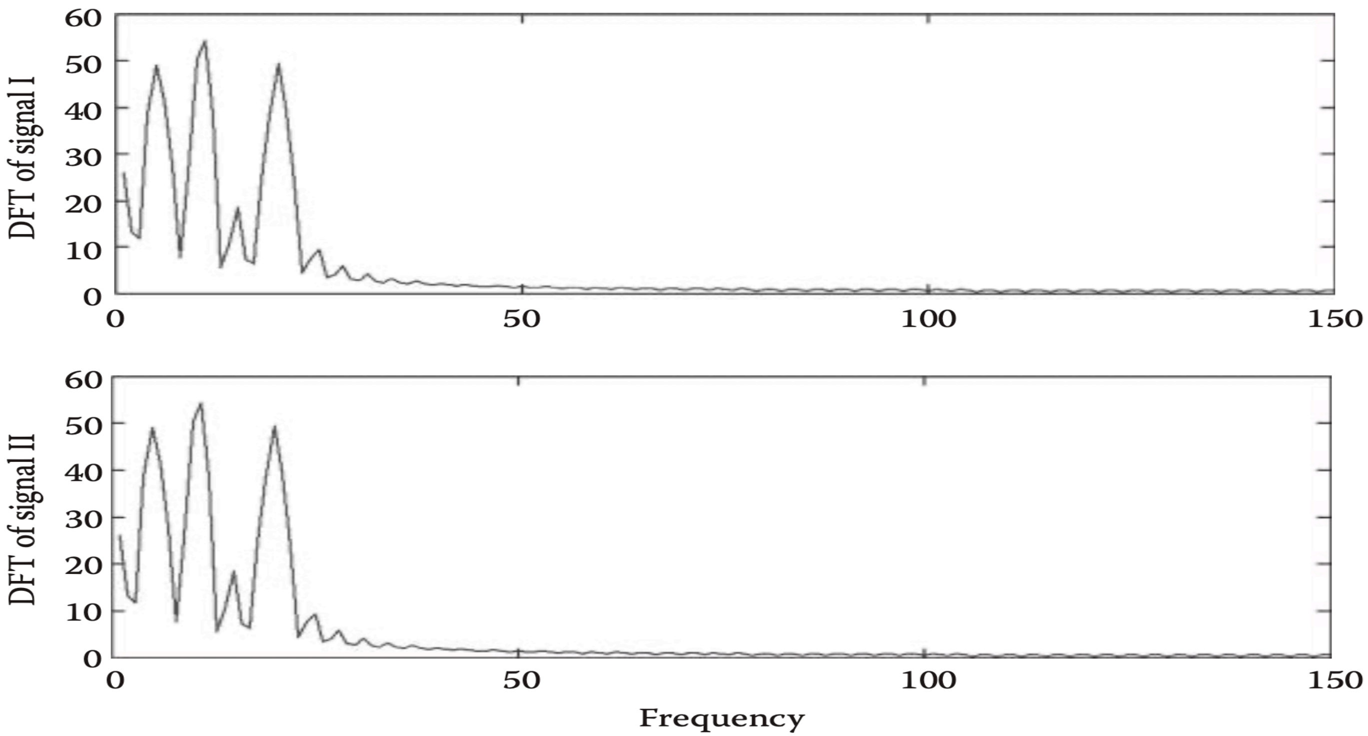


Two signals in time

From FT to STFT

- Concern over the use of the FT
  - Incapable of locating singularity

Example:



Magnitude of the DFT of the two signals

From FT to STFT

- Short Time Fourier Transform (STFT)
  - For a signal x(t), the STFT is defined as:

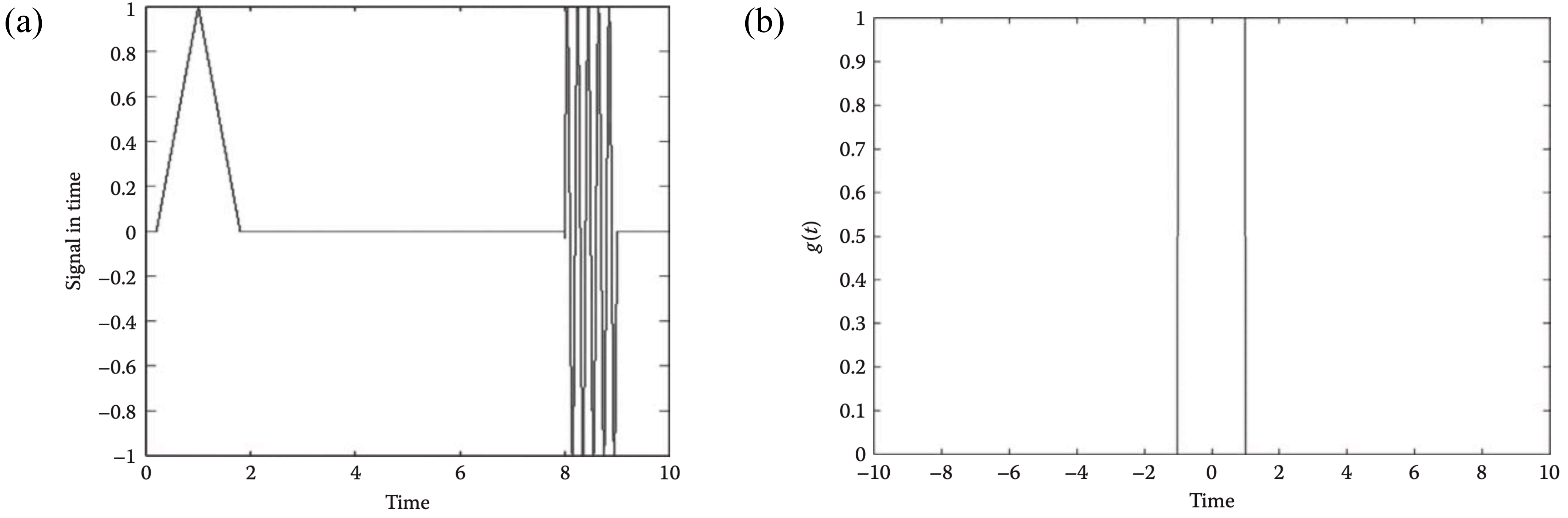
$$X_{\mathrm{STFT}}(a, f) = \int_{-\infty}^{+\infty} x(t) g*(t - a) e^{-j2\mathrm{p}ft} dt$$

where g(t-a) is a shifted version of a time window (gate) g(t) that extracts a portion of the signal x(t). In other words, the gate g(t-a), having a limited time span, selects and extracts only a portion of the signal x(t) to be analyzed by the FT. This time windos is often a real-time function, and, therefore,

$$g*(t - a) = g(t - a)$$

From FT to STFT

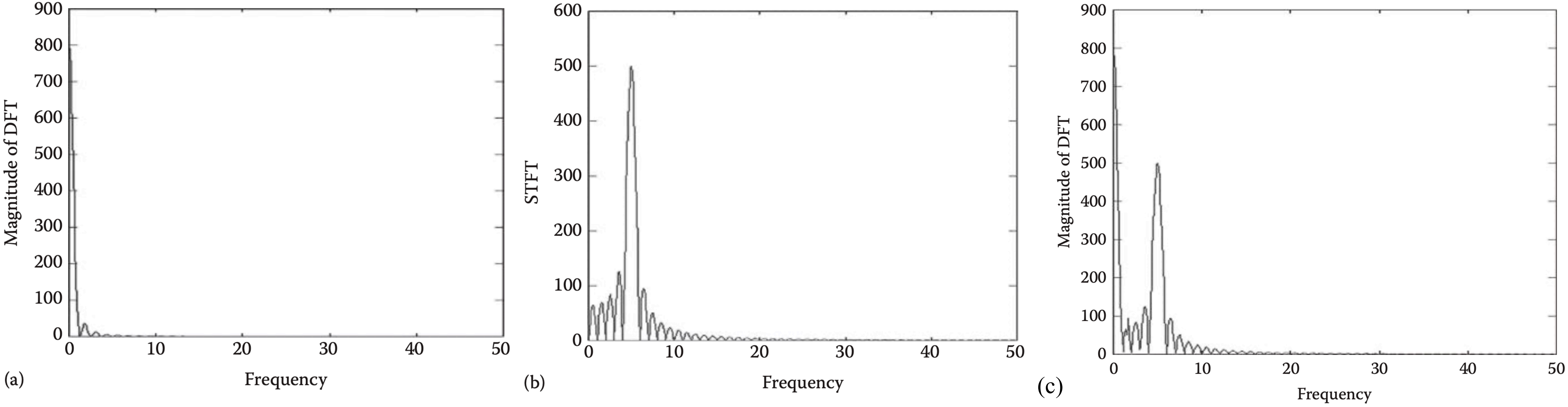
- Short Time Fourier Transform (STFT)
  - Example:



(a) The test signal. (b) The gate window g(t)

From FT to STFT

- Short Time Fourier Transform (STFT)
  - Example:



(a)Magnitude of STFT for a=1. (b)The magnitude of STFT for a=8 (c) FT of example signal.

From FT to STFT

- Shortcomings of STFT
  - The choice of the window length. A too short window may not capture the entire duration of an event, while a too long window may capture two or more events in the same shift.
  - The nature of the basis functions used in the FT, i.e, complex exponentials. The term $e^{-j2\pi ft}$ describes sinusoidal variations in the real and complex spaces. Such sinusoidal functions exist in all times and are not limited in time span or duration.

Wavelet Transform

- Take a closer look at the basic definition of “frequency”
  - Since we will not be using periodic sinusoidal basis functions for the new transform, we need to think of a concept that replaces frequency. Time-limited basis functions are obviously not periodic, and therefore, we need to invent a new concept that can represent a concept similar to frequency.

Wavelet Transform

- Take a closer look at the basic definition of “frequency”
  - what interesting features are captured by frequency?
    - The harmonic relation among the basis signals is the fundamental concept of signal transformation and decomposition. Therefore, the relation among harmonics is something that we need to somehow represent by our new concept that will replace frequency.

Wavelet Transform

- Take a closer look at the basic definition of "frequency"
  - A observation about harmonics
    - Warping the time axis "t" allows us to obtain the harmonics from the original signal, for example, replacing the time axis "t" in the original signal with "2t" time axis results in the second harmonic. This is essentially "scaling" the signal in time to generate other basis functions. We claim here that the main characteristic of harmonic frequencies can be drawn from a more general concept that we call "scale ".

Wavelet Transform

- Take a closer look at the basic definition of “frequency”
  - The advantages of the new concept
    - Unlike frequency that is defined only for periodic signals, scale is equally applicable to nonperiodic signals
    - Using scale as a variable, the new transform, which will be based on time-limited basis function, can be meaningfully applied to both time-unlimited and time-limited signals

Wavelet Transform

- The defination of continuous wavelet transform (CWT)

$$W_{\Psi,X}(a,b) = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{+\infty} x(t)\Psi^*\left(\frac{t-b}{a}\right) dt, \quad a \neq 0$$

In this equation, which is also referred to as the CWT analysis equation, $\Psi(t)$ is a function with limited duration in time, b is the shifting parameter, and a is the scal- ing parameter (replacing frequency parameter f)

Wavelet Transform

- The defination of continuous wavelet transform (CWT)
  - Due to the central role of the function $\Psi(t)$ in generating the basis functions of the CWT, this function is often referred to as the mother wavelet.
  - A closer look at the definition of the mother wavelet tells that this function must be limited in duration and therefore looks like a decaying small wave.
  - All other basis functions are the shifted and scaled version of the mother wavelet. A mother wavelet is continuously shifted and scaled to create all basis functions in CWT.

Wavelet Transform

- The defination of inverse continuous wavelet transform (ICWT)

$$x(t) = \frac{C_{\Psi}^{-1}}{a^2} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W_{\Psi,X}(a,b)\Psi\left(\frac{t-b}{a}\right) da\, db, \qquad a \neq 0$$

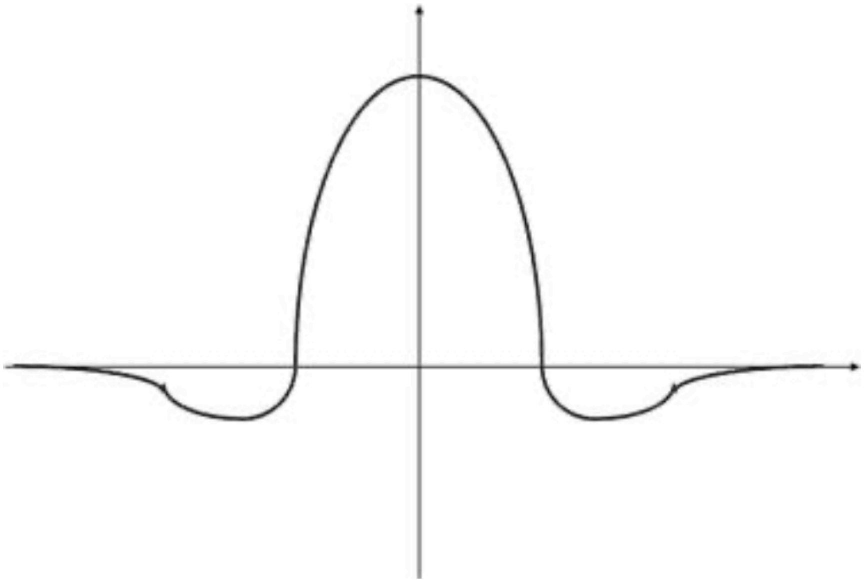
In this equation, C_{ψ}^{-1} is a constant whose value depends on the exact choice of the mother wavelet Ψ(t) As can be seen, while the CWT equation is a single integral, the ICWT is a double integral based on two dummy variables a and b.

Wavelet Transform

- Mother wavelets
  - Every choice of the mother wavelet gives a particular CWT, and as a result, we are dealing with infinite number of transformations under the same name CWT.
  - Any choice of the mother wavelet gives certain unique properties that make the resulting transformation a suitable choice for a particular task.

Wavelet Transform

- Mother wavelets
  - Mexican hat wavelet



The reason this mother wavelet is called Mexican hat (or sombrero) is evident from its waveform. The wave- form represented by this signal is the general shape of the most popular mother wavelets.

Wavelet Transform

- Mother wavelets
  - Daubechies (dbX) wavelets are among the most popular mother wavelets that are commonly used in signal and image processing.
  - The index “X” in dbX identifies the exact formulation of the function.
  - More complex mother wavelet may be needed to analyze more complex signals. One would limit X to 10 or 15 for ECG while 30 for speech signals.

Wavelet Transform

- Mother wavelets
  - Other popular mother wavelets: Coiflets, Symlets, Morlet, Meyer, e.t.c.

  - How to choose a mother wavelet for a particular application:
    - complex mother wavelets are needed for complex signals (as discussed earlier).
    - the mother wavelet that resembles the general shape of the signal to be analyzed would be a more suitable choice.

Wavelet Transform

- One-Dimensional discrete wavelet transform

$$W_{jk} = \int_{-\infty}^{+\infty} x(t)\Psi_{jk}^{*}(t)dt$$

where $\Psi_{jk}(t) = \dfrac{1}{\sqrt{a_{jk}}}\Psi\left(\dfrac{t - b_{jk}}{a_{jk}}\right) = a_0^{-j/2}\,\Psi\left(a_0^{-j}t - kT\right)$

$$a_{jk} = a_0^j, \quad b_{jk} = ka_0^jT$$

T is the sampling time

a_0 is a positive nonzero constant

Wavelet Transform

- One-Dimensional discrete wavelet transform

$$x(t) = c\sum_{j=0}^{N-1}\sum_{k=0}^{M-1} W_{jk}\Psi_{jk}(t)$$

In this equation, c is a constant that depends on the exact choice of the mother wavelet.
The interesting thing about this equation is the fact that we can reconstruct the continuous signal directly from a set of discrete coefficients.

Wavelet Transform

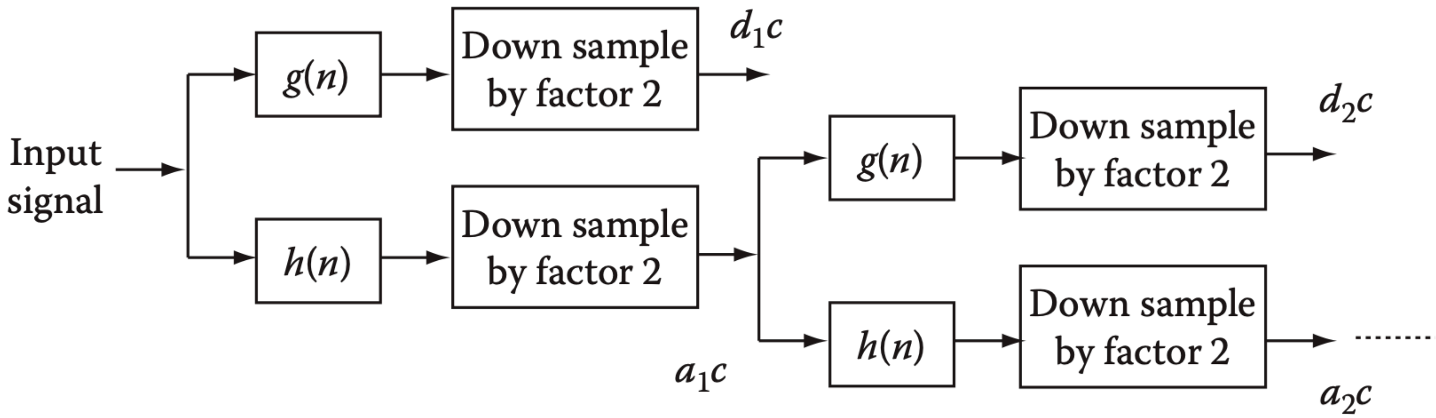
- How to choose the number of basis functions for a given signal?
  - A frame is a set of basis functions that can be used to decompose a signal. This set can be minimal or nonminimal. If the number of basis functions in the frame is minimal and any other frame would need the same number or more basis functions, the frame is called a basis.
  - Define $E_x = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt$. For a frame formed based on the functions $\psi_{jk}(t)$, it can be proved that: $A.E_x \le \sum_{j=0}^{N-1} \sum_{k=0}^{M-1} |W_{jk}|^2 \le B.E_x$ where A, B are bounded positive values.
  - A frame is a basis when $E_x = \sum_{j=0}^{N-1} \sum_{k=0}^{M-1} |W_{jk}|^2$

Wavelet Transform

- Discrete wavelet transform on discrete signals
  - In almost all practical applications, signals are formed of discrete measurements, and therefore in practice we normally deal with sampled signals.

Wavelet Transform

- Discrete wavelet transform on discrete signals
  - How to form basis sets systematically? Mallat pyramidal algorithm(or quadrature mirror filter, QMF)
    - The interesting feature of this method is the fact that the method relies only on the choice of a digital low-pass filter $h(n)$, and the restrictions on $h(n)$ are so relaxed.



$$g(n) = h(2N - 1 - n)$$

Wavelet Transform

- Discrete wavelet transform on discrete signals
  - What's the mother wavelet of the QMF algorithm?

$$\Psi(n) = \sum_{i=0}^{N-1} g(i)\Phi(2n-i)$$

where

$$\Phi(n) = \sum_{i=0}^{N-1} h(i)\Phi(2n-i)$$

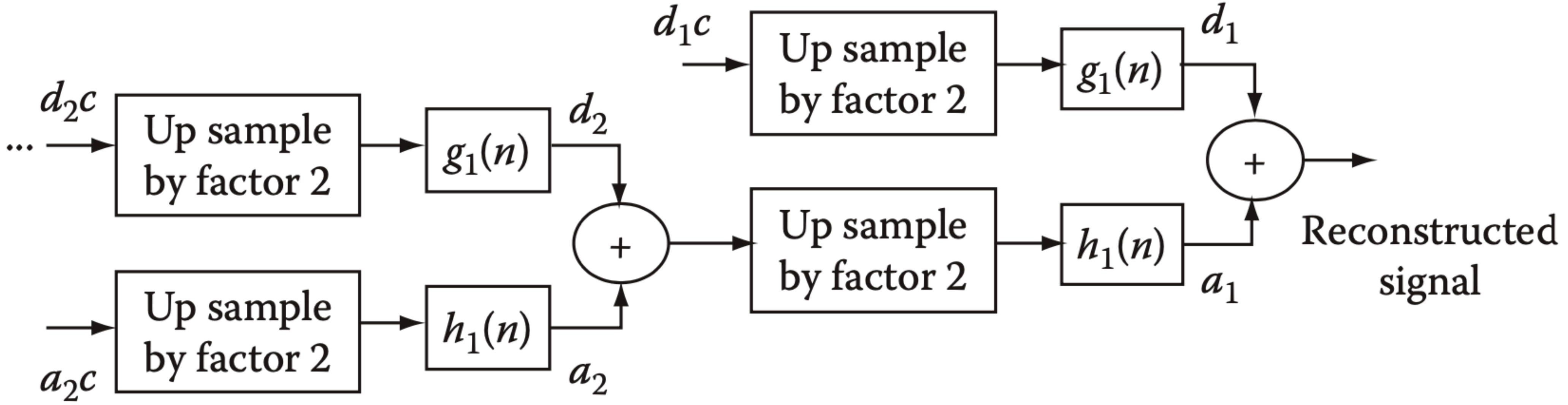
Wavelet Transform

- Discrete wavelet transform on discrete signals
  - How many decomposition levels are needed for a suitable transform ?

Continuing decomposi- tion until the highest known frequencies in the signal of interest are extracted and identified Loosely speaking, if one needs to have more detailed decomposition of the signal in higher frequencies, he or she would need to calculate higher levels of decomposition This simply would allow more specific description of high-frequency components of a signal

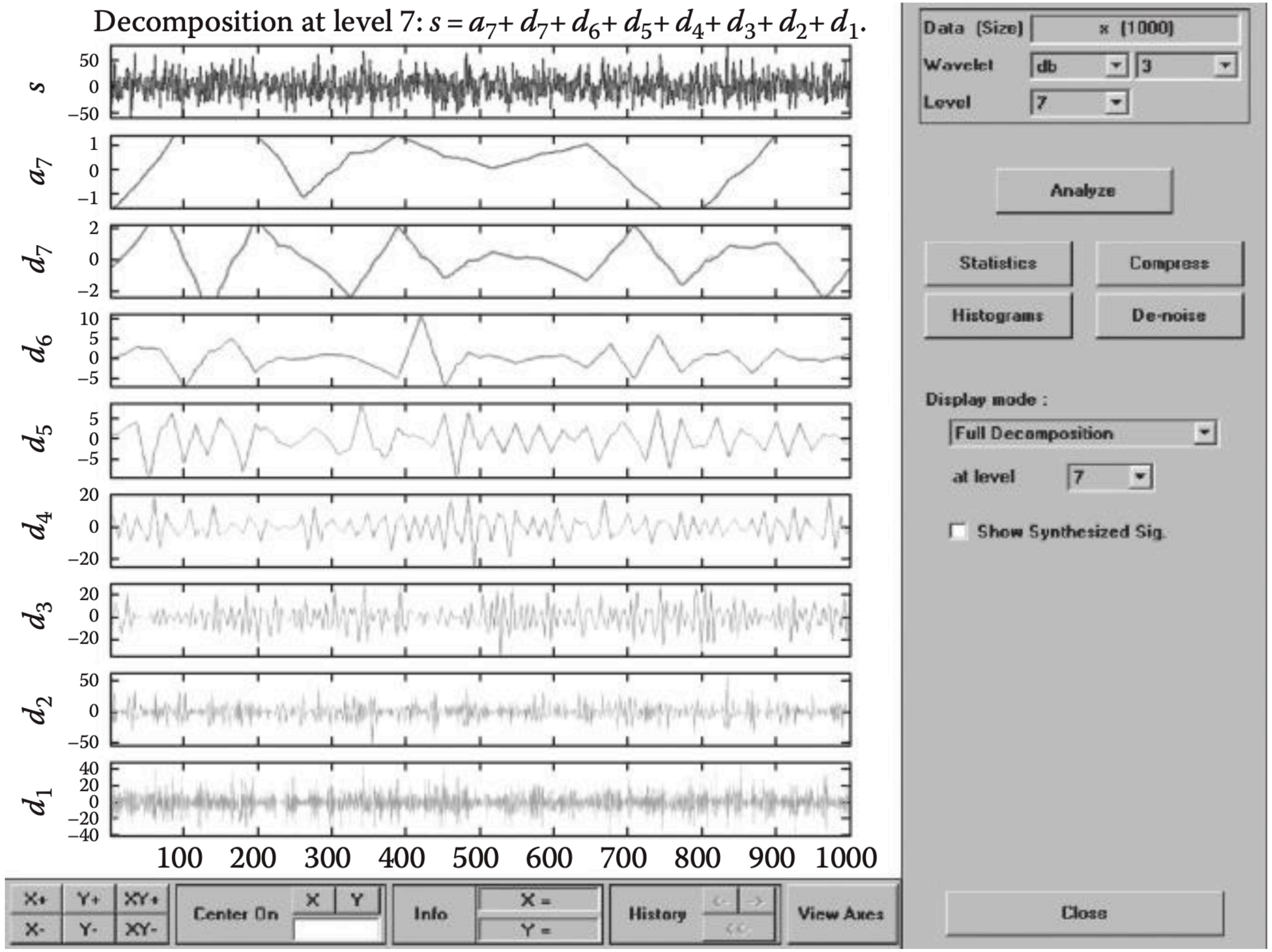
Wavelet Transform

- Discrete wavelet transform on discrete signals
  - The IDWT using QMF algorithm



$$h_1(n) = (-1)^{1-n} h(1-n) \qquad g_1(n) = h_1(2N-1-n)$$

Example

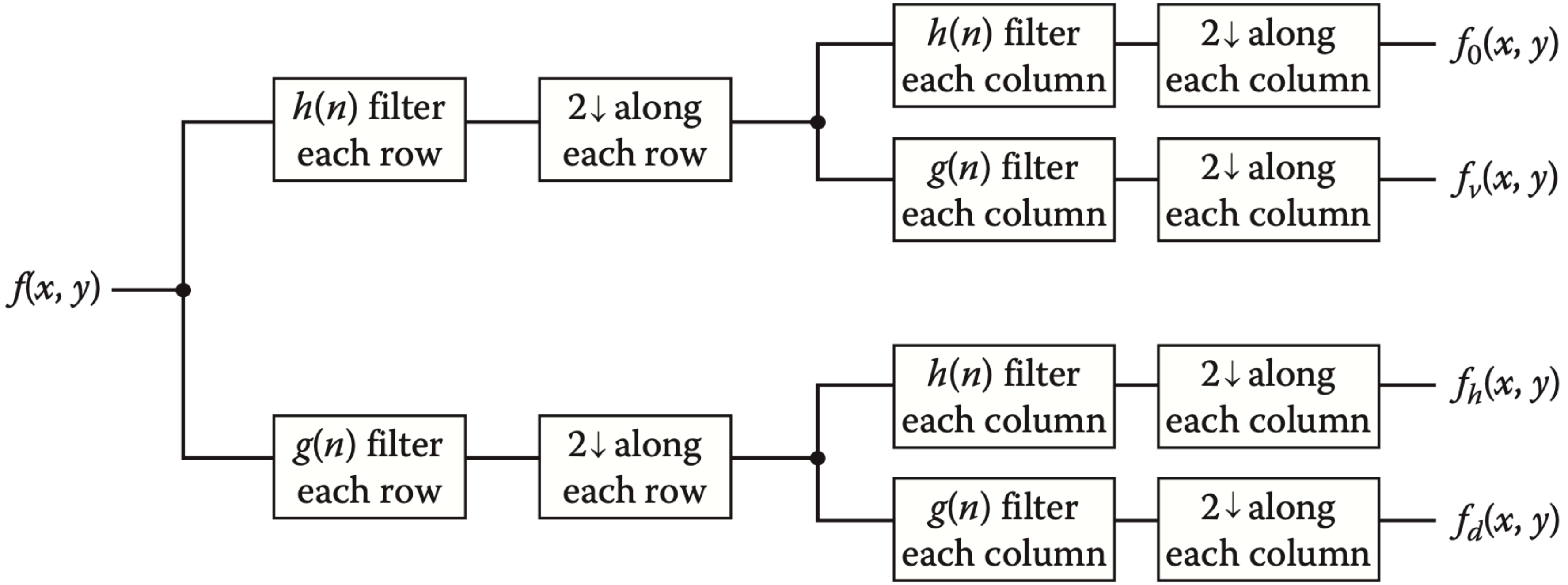


Decomposition and reconstruction of an EEG signal at different resolutions.

The strength (power) of the reconstructed signals at different levels has important physiological meanings and applications. For instance, in an EEG signal, a very strong low-frequency component (second graph from the top) identifies that the patient might be asleep or about to fall asleep. This will be further discussed in a chapter dedicated to EEG.

Wavelet Transform

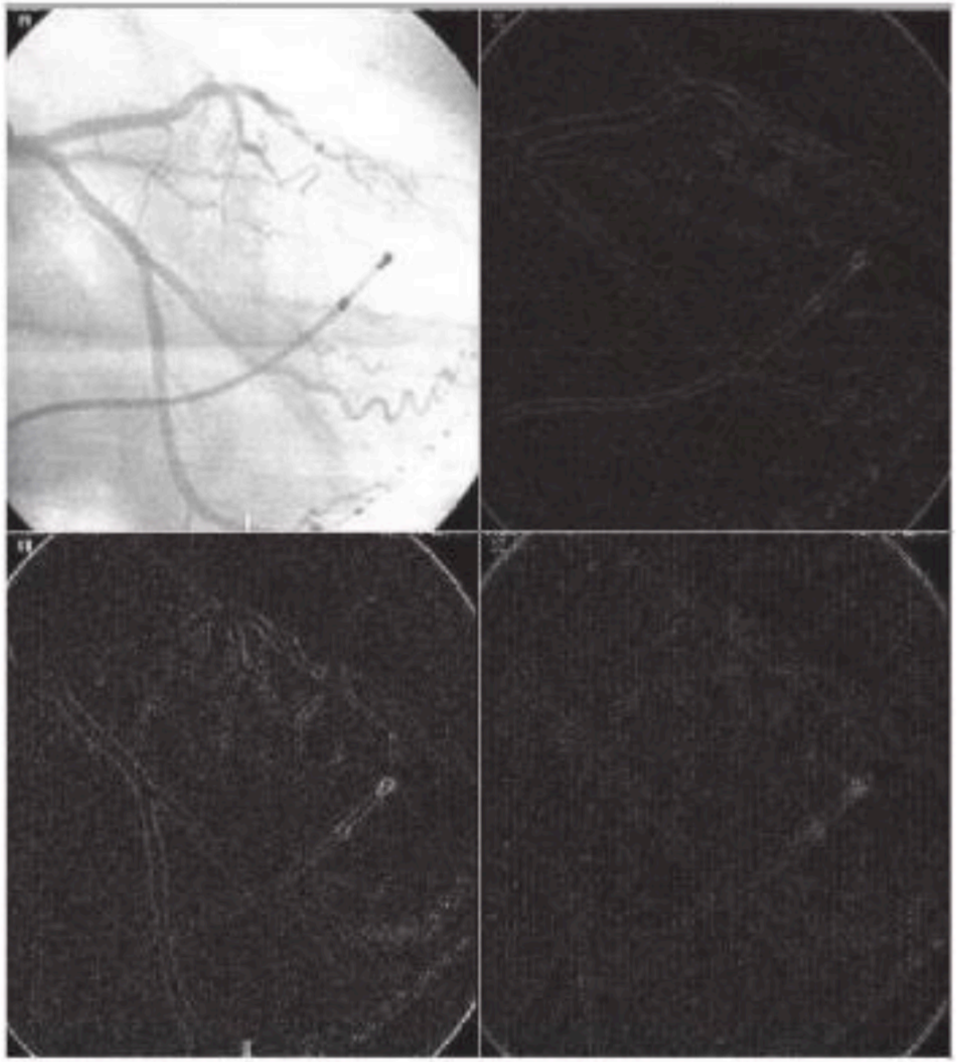
- Two-dimensional wavelet transform
  - Two-dimensional discrete wavelet transform



Schematic diagram of 2-D DWT

Example

Image selection



Decomposition at level 1

Decomposition of an image to the first level using DWT.

Wavelet Transform

- Main applications of DWT
  - Filtering and denoising
  - Compression
  - Extraction of scale-based features

Wavelet Transform

- Main applications of DWT
  - Filtering and denoising
    - As mentioned earlier, it is often the case that noise exists in high frequencies (i e , low scales) of signals and images For instance, electromagnetic drifts that appear in the wires and electrodes are high-frequency noise   Such a noise appears in almost all biomedical signals such as electroencephalogram and electrocardiogram.
    - Two main types of DWT-based denoising are used in signal processing: hard thresholding and soft thresholding

$$d_{jk}^{hard} = \begin{cases} d_{jk} & \text{if } |d_{jk}| > \mathbf{x} \\ 0 & \text{Otherwise} \end{cases}$$

$$d_{jk}^{soft} = \begin{cases} d_{jk} - \mathbf{x} & \text{if } d_{jk} > \mathbf{x} \\ 0 & \text{if } |d_{jk}| \le \mathbf{x} \\ d_{jk} + \mathbf{x} & \text{if } d_{jk} < -\mathbf{x} \end{cases}$$

Wavelet Transform

- Main applications of DWT
  - Compression
    - In many biomedical applications, long signals and large images are created The stor- age of all these signals and images creates a serious issue The main objective here is to design techniques to reduce the size of a signal or image without compromising the information contained in the signal or image
    - Rule for compression:
      - Eliminating small coefficients.
      - Eliminating too-high-frequency components.
      - Eliminating too-low-frequency components.

Multiresolution Analysis

Background

- Objects are formed by connected regions of similar texture and intensity levels.

- If the objects are small in size or low in contrast, we normally examine them at high resolutions; if they are large in size or high in contrast, a coarse view is required.

Multiresolution processing

Background

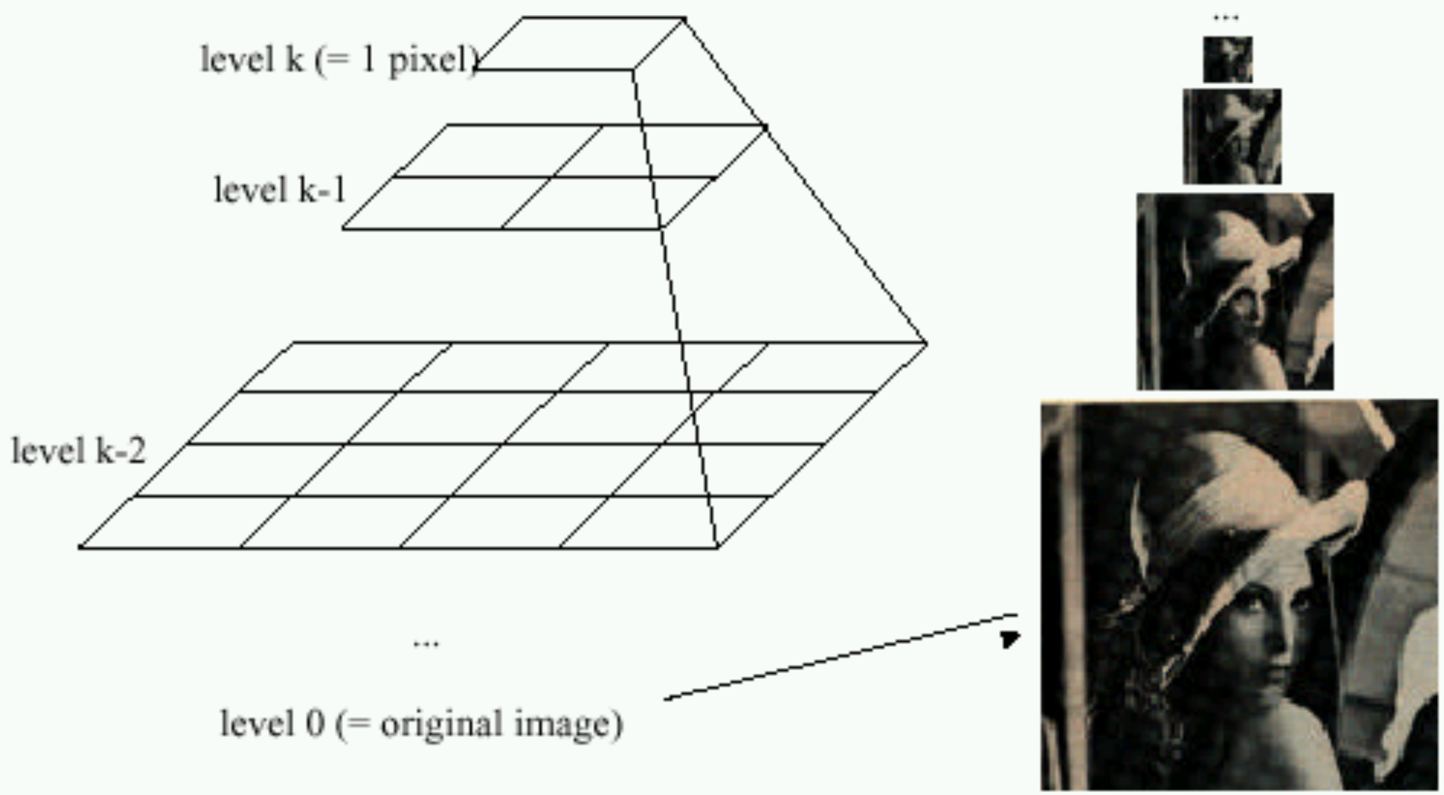
- Image Pyramids

  An image pyramid is a collection of decreasing resolution images arranged in the shape of a pyramid

  The base of the pyramid contains a high-resolution representation of the image being processed; the apex contains a low-resolution approximation.

Image Pyramids

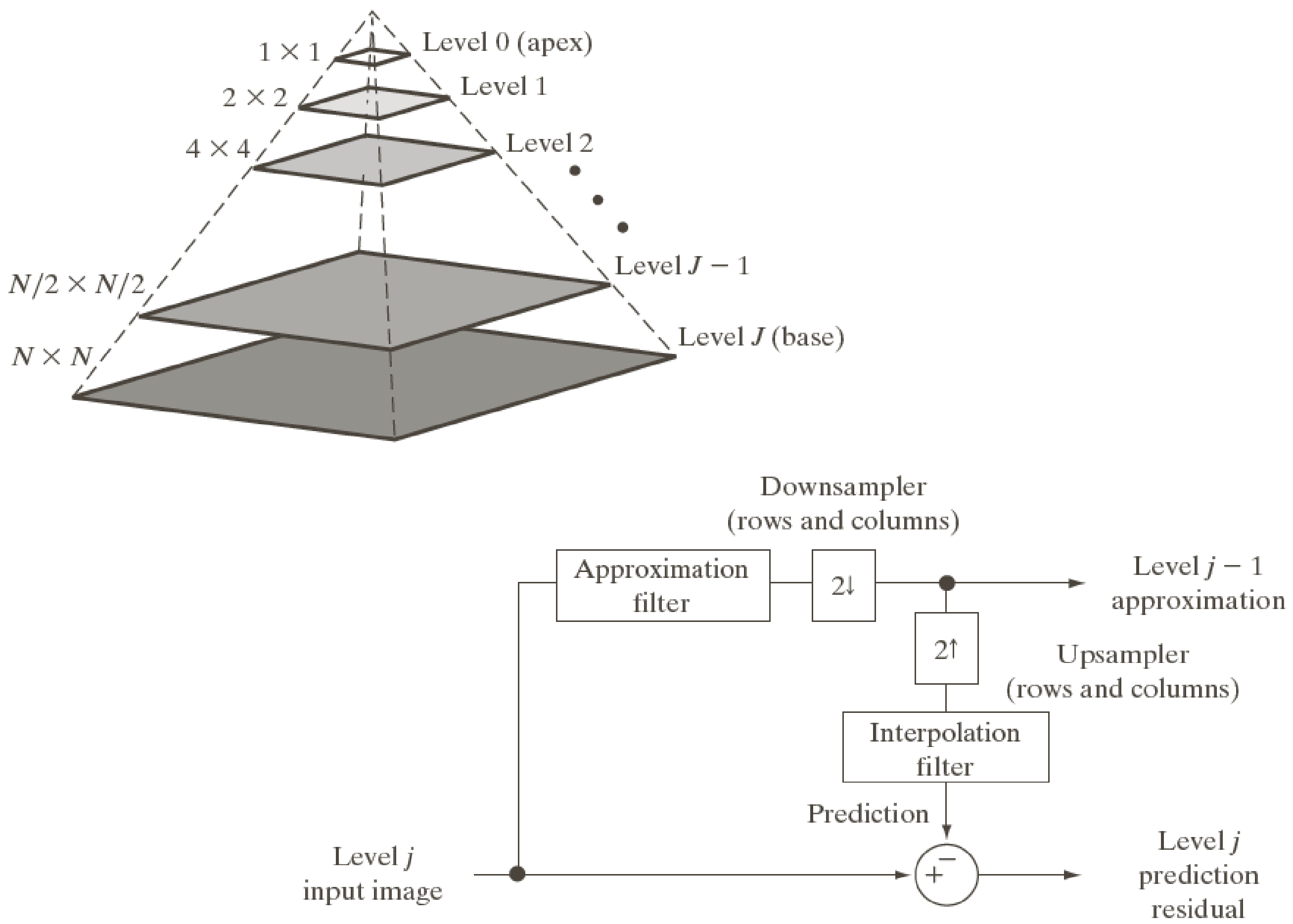
Idea: Represent NxN image as a "pyramid" of 1×1, 2×2, 4×4,…, $2^k \times 2^k$ images (assuming $N=2^k$)



Known as a **Gaussian Pyramid** [Burt and Adelson, 1983]

- In computer graphics, a *mip map* [Williams, 1983]
- A precursor to *wavelet transform*

Image Pyramids



a b

FIGURE 7.2
(a) An image pyramid. (b) A simple system for creating approximation and prediction residual pyramids.

Image Pyramids

- Gaussian Pyramid
  - Approximation pyramid
- Laplacian Pyramid
  - Prediction residual pyramid

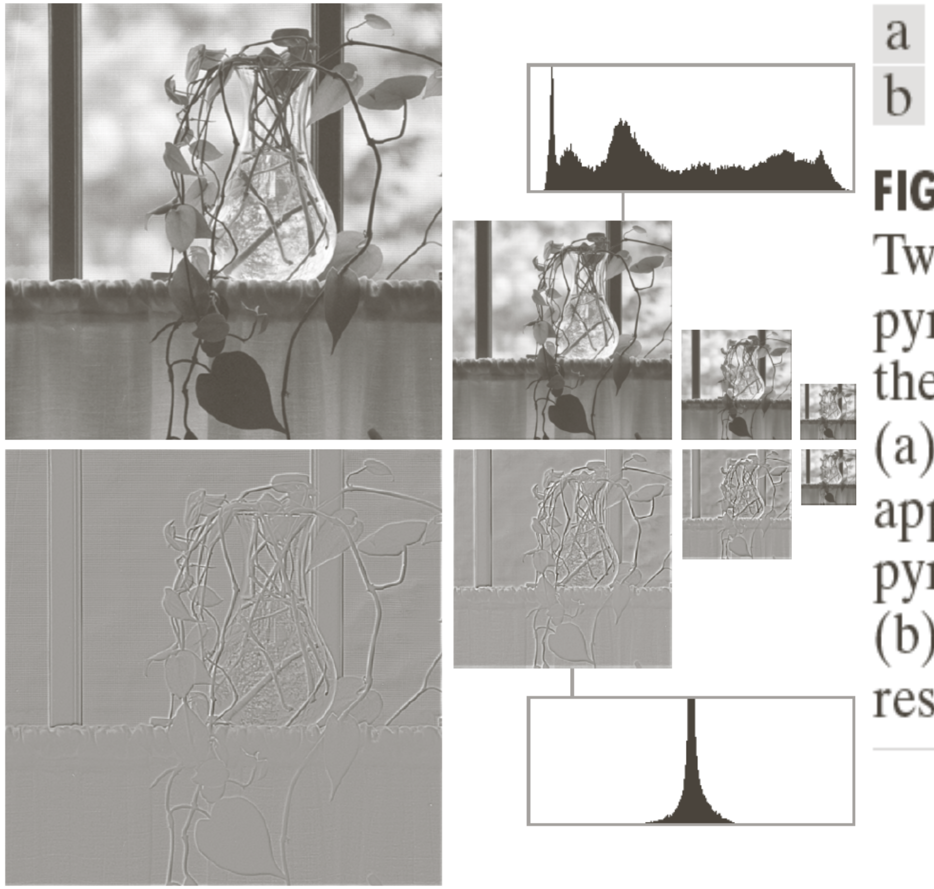


FIGURE 7.3 Two image pyramids and their histograms: (a) an approximation pyramid; (b) a prediction residual pyramid.

Gaussian pyramid construction

Step:

Repeat {

- Filter
- Subsample

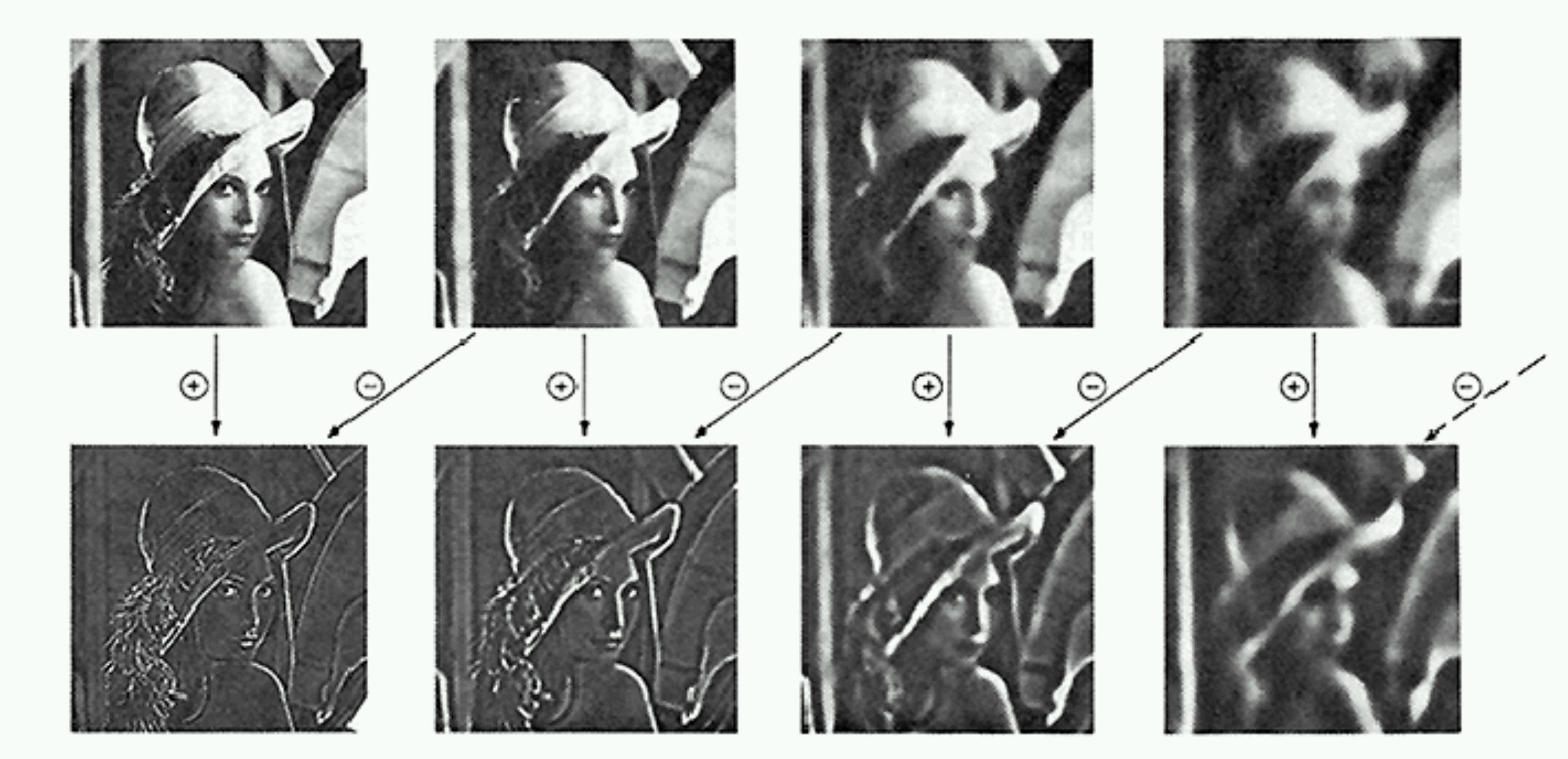
} Until minimum resolution reached

- can specify desired number of levels (e.g., 3-level pyramid)

The whole pyramid is only 4/3 the size of the original image!

Laplacian pyramid construction

- Created from Gaussian pyramid by subtraction



Gaussian Pyramid

Laplacian pyramid

What are they good for?

- Improve Search
  - Search over translations
    - Like homework
    - Classic coarse-to-fine strategy
  - Search over scale
    - Template matching
    - E.g. find a face at different scales
- Precomputation
  - Need to access image at different blur levels
  - Useful for texture mapping at different resolutions
- Image Processing
  - Editing frequency bands separately
  - E.g. image blending… next time!

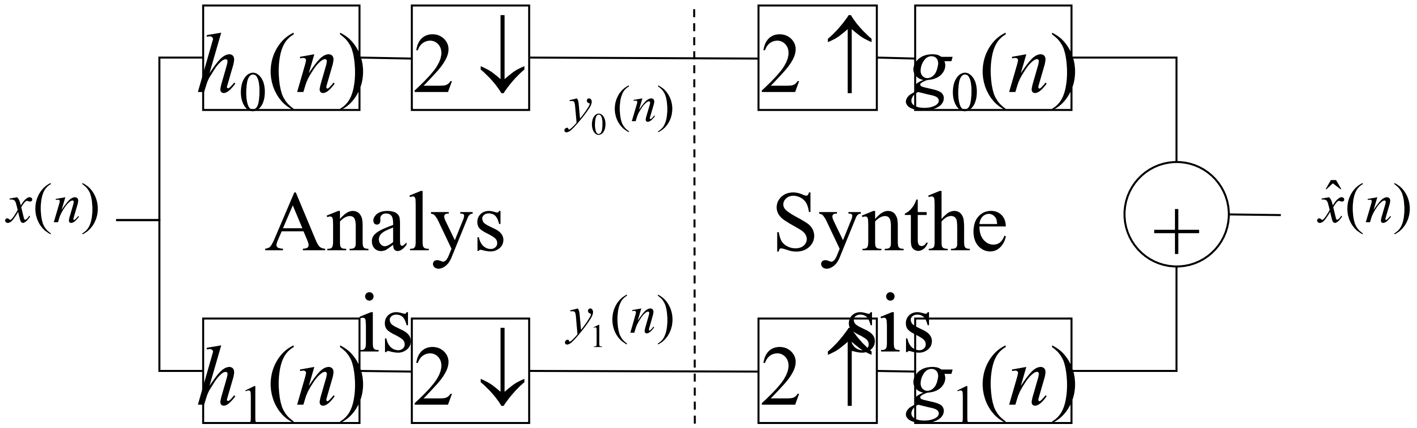
Background

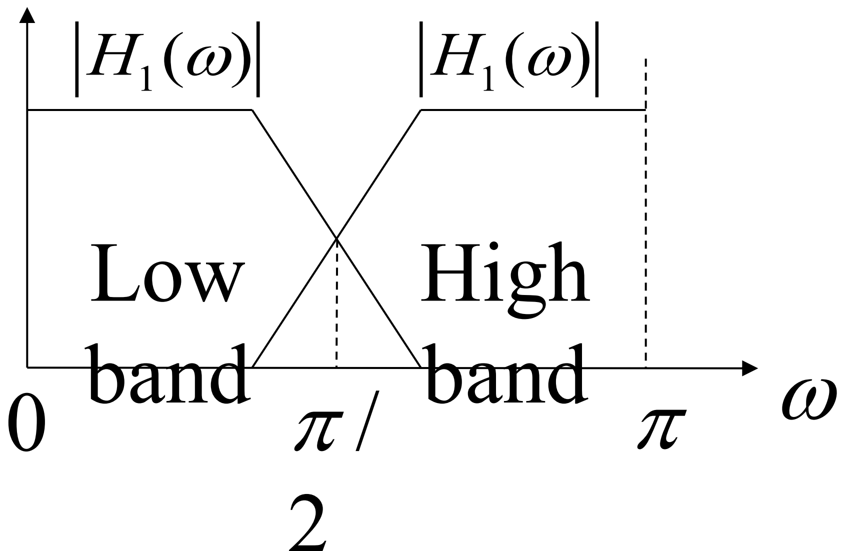
- Subband Coding

  Another important imaging technique with ties to multiresolution analysis is subband coding

  In subband coding, an image is decomposed into a set of bandlimited components, called subbands

Subband Coding





Subband Coding

- For perfect reconstruction, the impulse responses of the synthesis and analysis filters must be related in one of the following two ways:

$$g_0(n) = (-1)^n h_1(n)$$
$$g_1(n) = (-1)^{n+1} h_0(n)$$

- or

$$g_0(n) = (-1)^{n+1} h_1(n)$$
$$g_1(n) = (-1)^n h_0(n)$$

Subband Coding

- The impulse responses of the synthesis and analysis filters can be shown to satisfy the following biorthogonality condition

$$\langle h_i(2n-k), g_j(k) \rangle = \delta(i-j)\delta(n), \quad i,j=\{0,1\}$$

- Of special interest in subband coding- and in the development of the fast wavelet transform- are filters that move beyond biorthogonality and require

$$\langle g_i(n), g_j(n+2m) \rangle = \delta(i-j)\delta(m), \quad i,j=\{0,1\}$$

- which defines orthonormality for perfect reconstruction filter banks.

Subband Coding

- Orthonormal filters can be shown to satisfy the following two conditions

$$g_1(n) = (-1)^n g_0(K_{even} - 1 - n)$$

$$h_i(n) = g_i(K_{even} - 1 - n), \quad i = \{0, 1\}$$

Subband Coding

- 1-D orthonormal and biorthogonal filters can be used as 2-D separable filters for the processing of images.

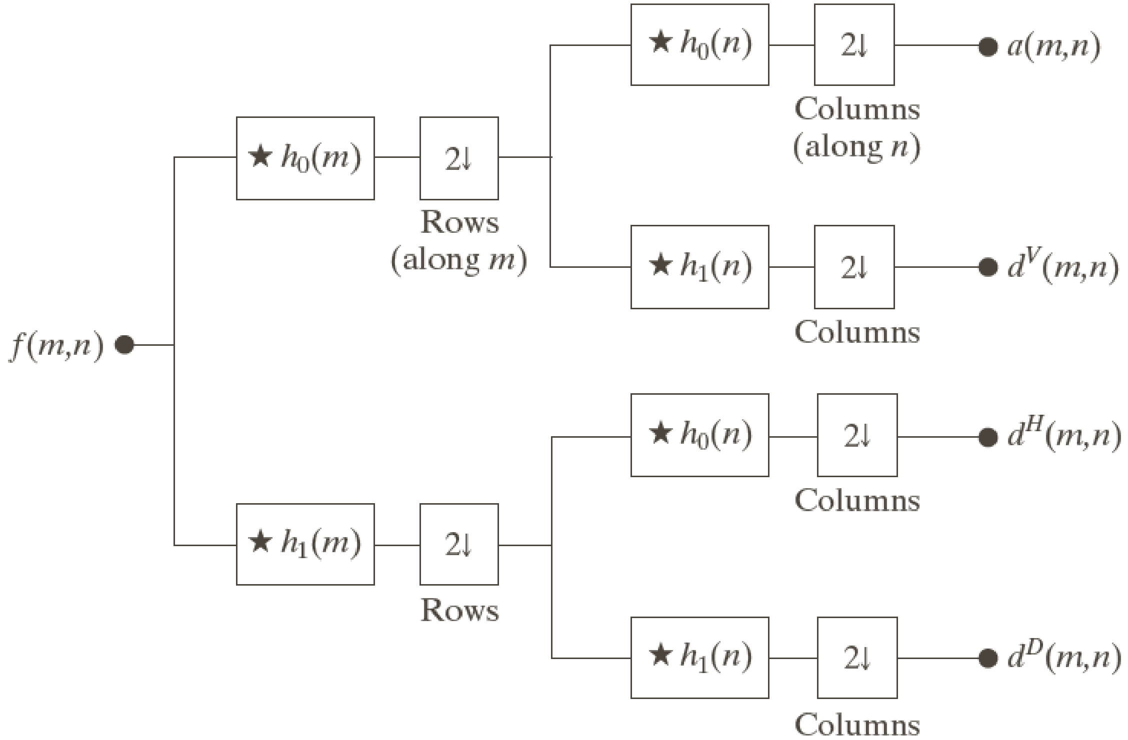


FIGURE 7.7 A two-dimensional, four-band filter bank for subband image coding.

Subband Coding

Example 7.2 a four-band subband coding of vase

n	$g_0(n)$
0	0.23037781
1	0.71484657
2	0.63088076
3	−0.02798376
4	−0.18703481
5	0.03084138
6	0.03288301
7	−0.01059740

TABLE 7.1 Daubechies 8-tap orthonormal filter coefficients for $g_0(n)$ (Daubechies [1992]).

Subband Coding

Example 7.2 a four-band subband coding of vase

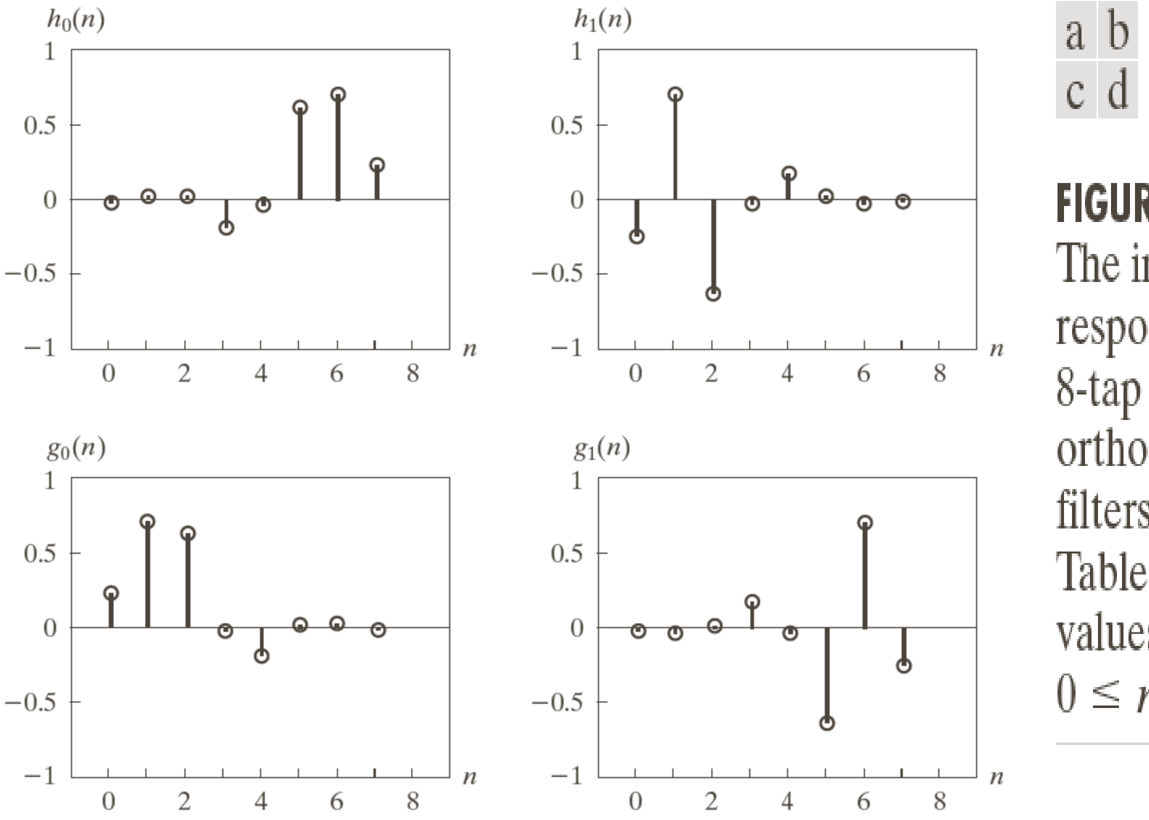


FIGURE 7.8 The impulse responses of four 8-tap Daubechies orthonormal filters. See Table 7.1 for the values of $g_0(n)$ for $0 \leq n \leq 7$.

Subband Coding

Example 7.2 a four-band subband coding of vase

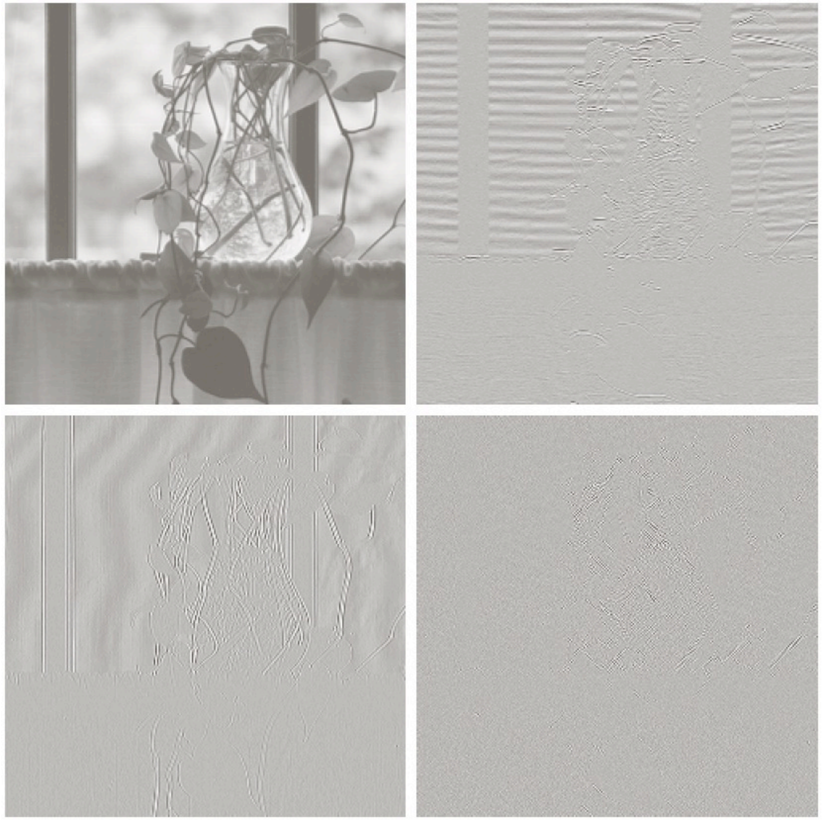


FIGURE 7.9 A four-band split of the vase in Fig. 7.1 using the subband coding system of Fig. 7.7. The four subbands that result are the (a) approximation, (b) horizontal detail, (c) vertical detail, and (d) diagonal detail subbands.

Background

- The Haar Transform
  - Haar transform is a special wavelet transform
  - Its basis functions are the oldest and simplest orthonormal wavelets
  - Haar transform can be expressed in a matrix form

$$T = HFH^T$$

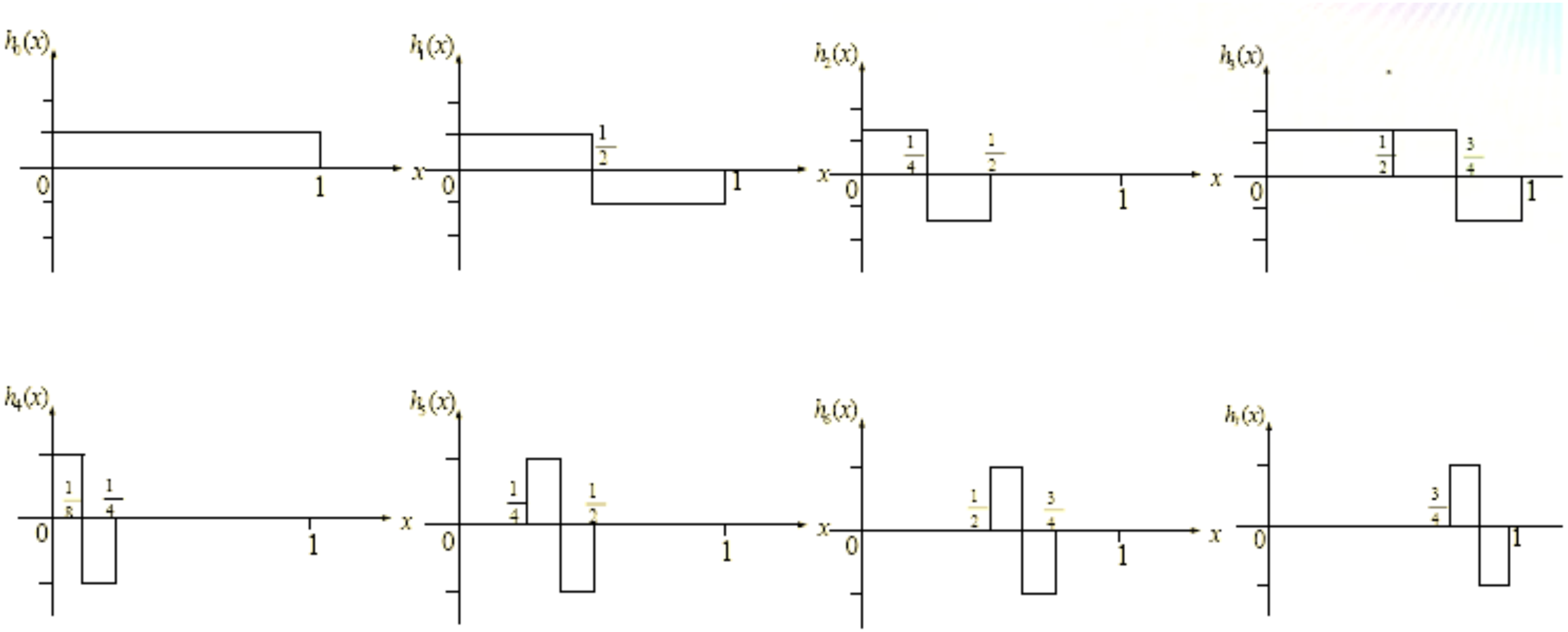
F is an $N \times N$ image matrix

H is an $N \times N$ transformation matrix

T is the resulting $N \times N$ transform

The Haar Transform

- Basis functions of Haar transform (continuous)



The Haar Transform

- All of the basis functions are rectangular impulse pairs except $h_0(z)$
- The impulse pairs have different width, height and positions
- Width of nonzero region is descending

$$1 \to \frac{1}{2} \to \frac{1}{4} \to \frac{1}{8} \to \ldots$$

- Height of nonzero region is ascending

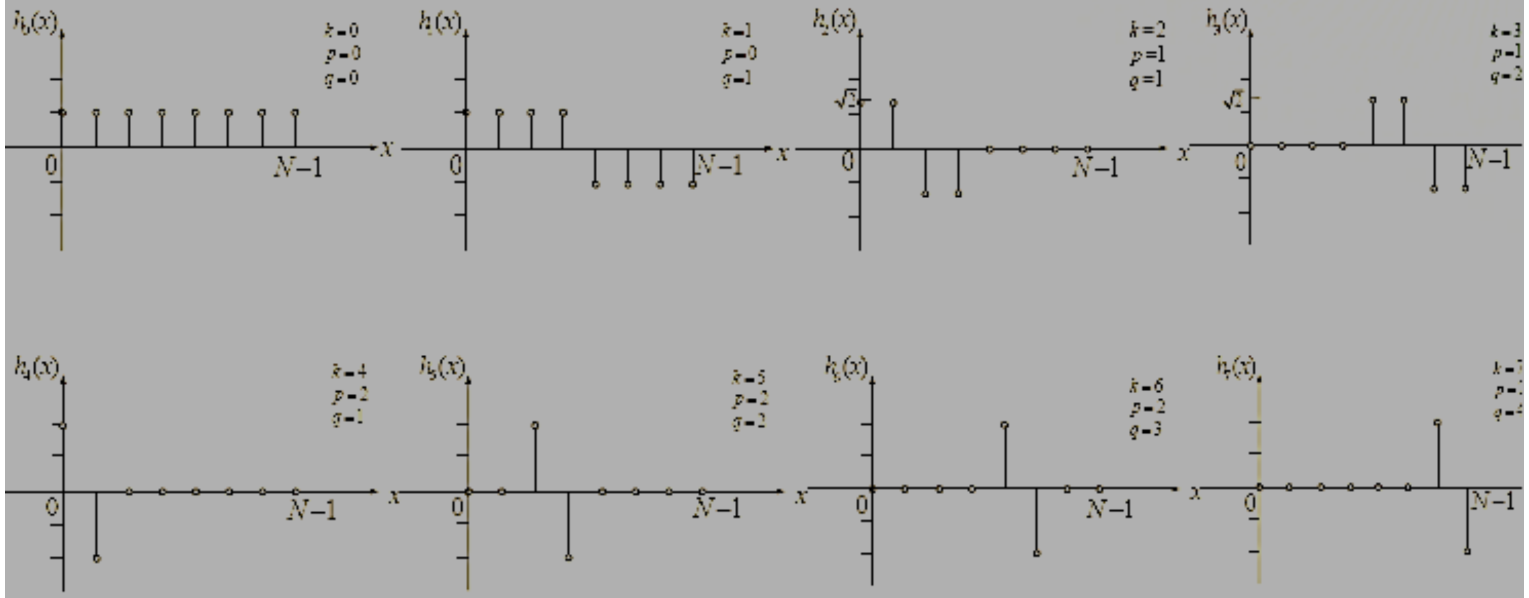
$$\frac{1}{\sqrt{N}} \to \frac{\sqrt{2}}{\sqrt{N}} \to \frac{2}{\sqrt{N}} \to \ldots$$

- The basis functions have the same characters as those of wavelet transform

The Haar Transform

- Basis functions of Haar transform (discrete)

$N = 8 \qquad h_k(x), k = 0 \sim 7$



$$k = 2^p + q - 1$$

$$2^p \leq k \text{ and } 2^{p+1} > k$$

The Haar Transform

- The ith row of an N by N Haar transformation matrix contains the elements of $\mathrm{h}_i(z)$ for $z = \frac{0}{N}, \frac{1}{N}, \frac{2}{N}, \cdots, \frac{N-1}{N}$

- 2×2 transformation matrix is

$$H_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

- 4×4 transformation matrix is

$$H_4 = \frac{1}{\sqrt{4}} 0 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

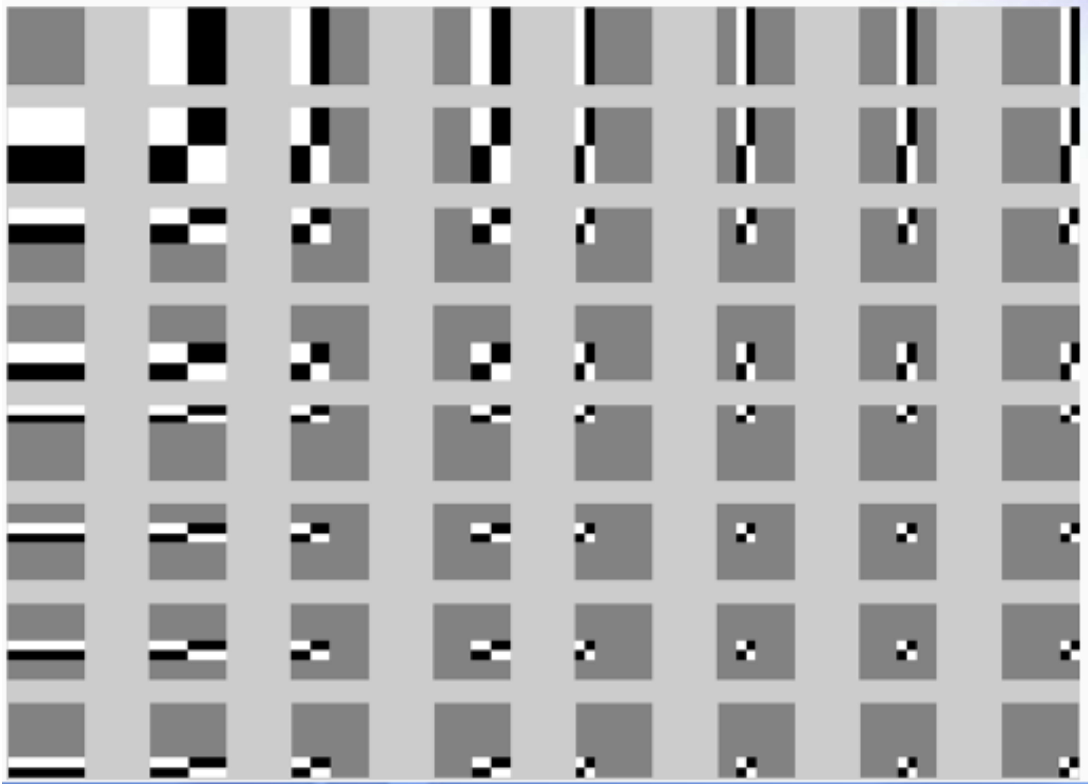
The Haar Transform

- 8×8 transformation matrix is

$$H_8 = \frac{1}{\sqrt{8}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix}$$

The Haar Transform

- Transformation kernel is separable
- For N = 8 , the basis functions are



The Haar Transform

- Example 7.3

Haar functions in a discrete wavelet transform.

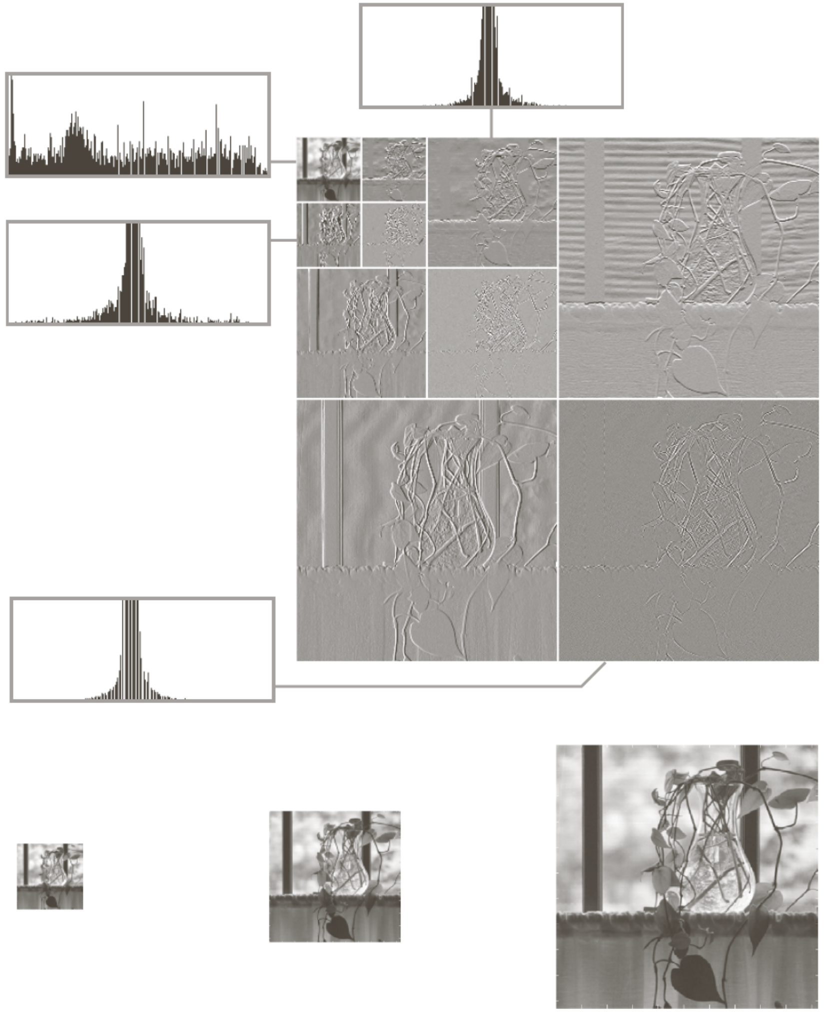


FIGURE 7.10 (a) A discrete wavelet transform using Haar $\mathbf{H}_2$ basis functions. Its local histogram variations are also shown. (b)–(d) Several different approximations (64×64, 128×128, and 256×256) that can be obtained from (a).

Glance at multiscale geometry analysis and sparse representation

Stable Analysis and Synthesis Operators

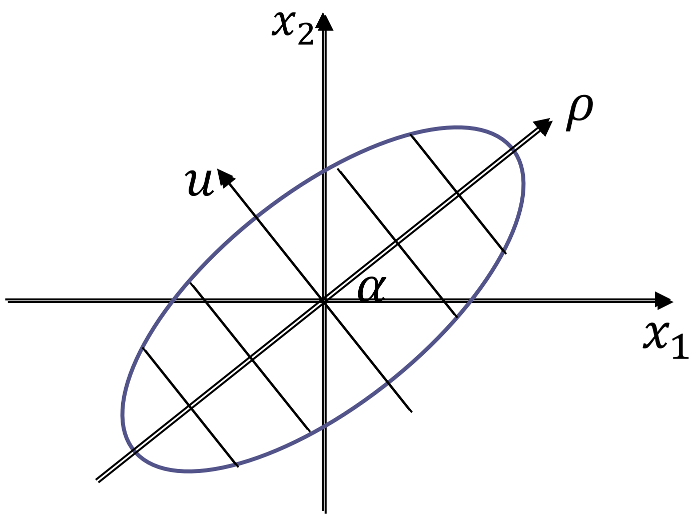
- To reveal geometric image properties, wavelet frames are constructed with mother wavelets having a direction selectivity, providing information on the direction of sharp transitions such as edges and textures.
- Wavelet frames yield high-amplitude coefficients in the neighborhood of edges, and cannot take advantage of their geometric regularity to improve the sparsity of the representation.
- Frames are potentially redundant and thus more general than bases, with a redundancy measured by frame bounds. They provide the flexibility needed to build signal representations with unstructured families of vectors.

Directional Wavelet Frames

Directional Vanishing Moment

- A directional wavelet $\psi^{\alpha}(x)$ with $x=(x_1,x_2)\in\mathbb{R}^2$ of angle α is a wavelet having p directional vanishing moments along any one-dimensional line of direction $\alpha+\frac{\pi}{2}$ in the plane:

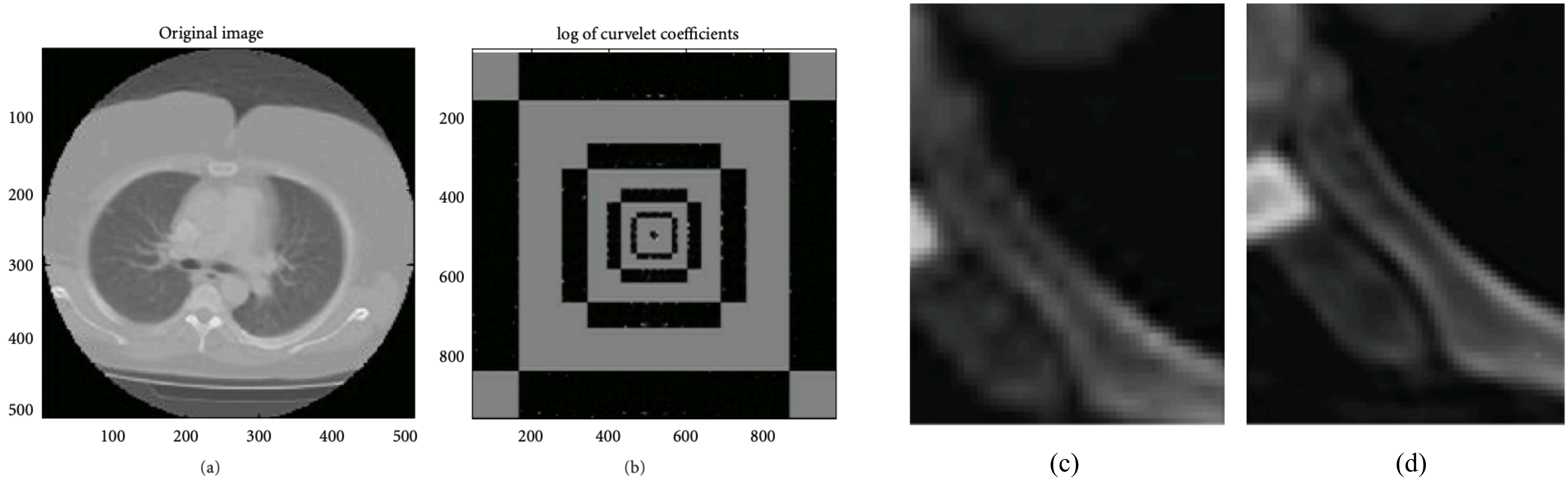
$\forall\rho\epsilon\mathbb{R}, \int\psi^{\alpha}(\rho\cos\alpha-u\sin\alpha\,,\rho\sin\alpha+u\cos\alpha)u^k du=0$ for $0\le k\le p$, but does not have directional vanishing moments along the direction α.



$$\begin{cases}x_1=\rho\cos\alpha-u\sin\alpha\\ x_2=\rho\sin\alpha+u\cos\alpha\end{cases}$$

- Directional wavelets may be derived by rotating a single mother wavelet $\psi(x_1,x_2)$ having vanishing moments in the horizontal direction, with a rotation operator R_{α} of angle α in $\mathbb{R}^2$.

Example



Clinical slice for the human chest from a CT scanner in spatial (a) and curvelet coefficients (b). Reconstruction of tomographic data. Wavelet domain (c), and curvelet domain (d).

AlZubi S, Islam N, Abbod M. Multiresolution analysis using wavelet, ridgelet, and curvelet transforms for medical image segmentation[J]. International journal of biomedical imaging, 2011, 2011.

What is a sparse linear model

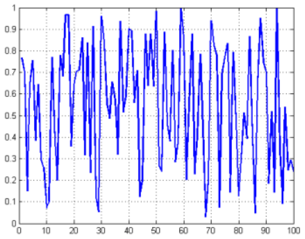
Consider a signal $\mathbf{x} \in \mathbb{R}^m$.

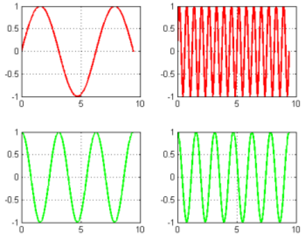
$\mathbf{D} = [\mathbf{d_1}, \dots, \mathbf{d_k}] \in \mathbb{R}^{m \times k}$ denotes a set of normalized elementary signals (atoms) used to decompose the signal.

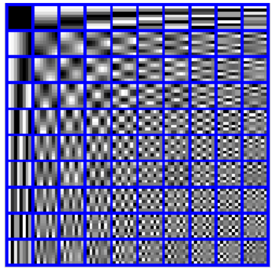
We call it **dictionary**.

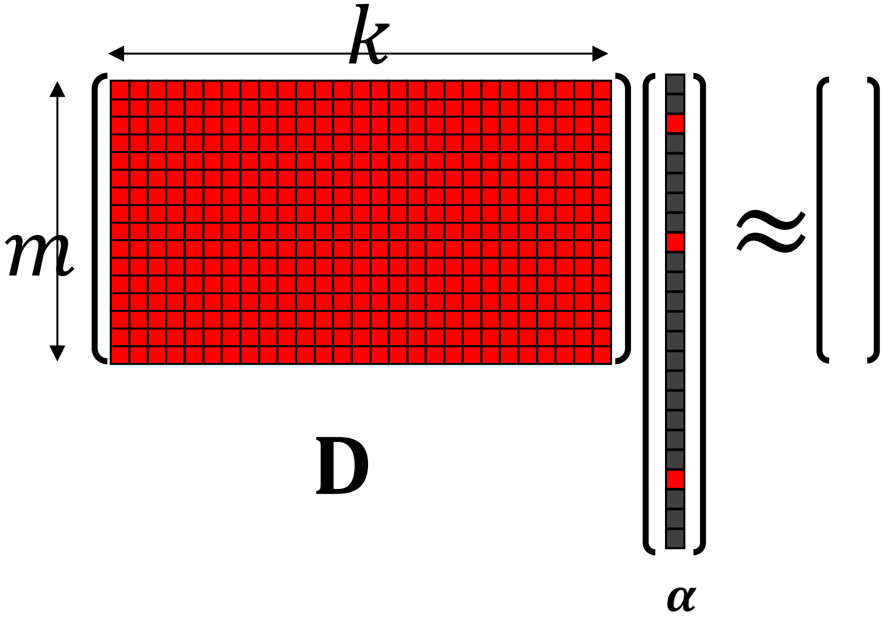
We say it admits a ***sparse approximation*** (a.k.a. sparse decomposition, sparse code) $\alpha \in \mathbb{R}^k$ over the dictionary $\mathbf{D}$, when one can find a linear combination of “a few” atoms from $\mathbf{D}$ that is “close” to the signal $\mathbf{x}$.

$$\mathbf{x} \approx \mathbf{D}\alpha, \text{ where } ||\alpha||_0 \ll m$$









Sparse Decomposition Problem

$$\min_{\alpha \in \mathbb{R}^k} \underbrace{\frac{1}{2}\|\mathbf{x} - \mathbf{D}\alpha\|_2^2}_{\text{Data fitting term}} + \underbrace{\lambda\psi(\alpha)}_{\text{Sparsity-inducing regularization}} \qquad (1)$$

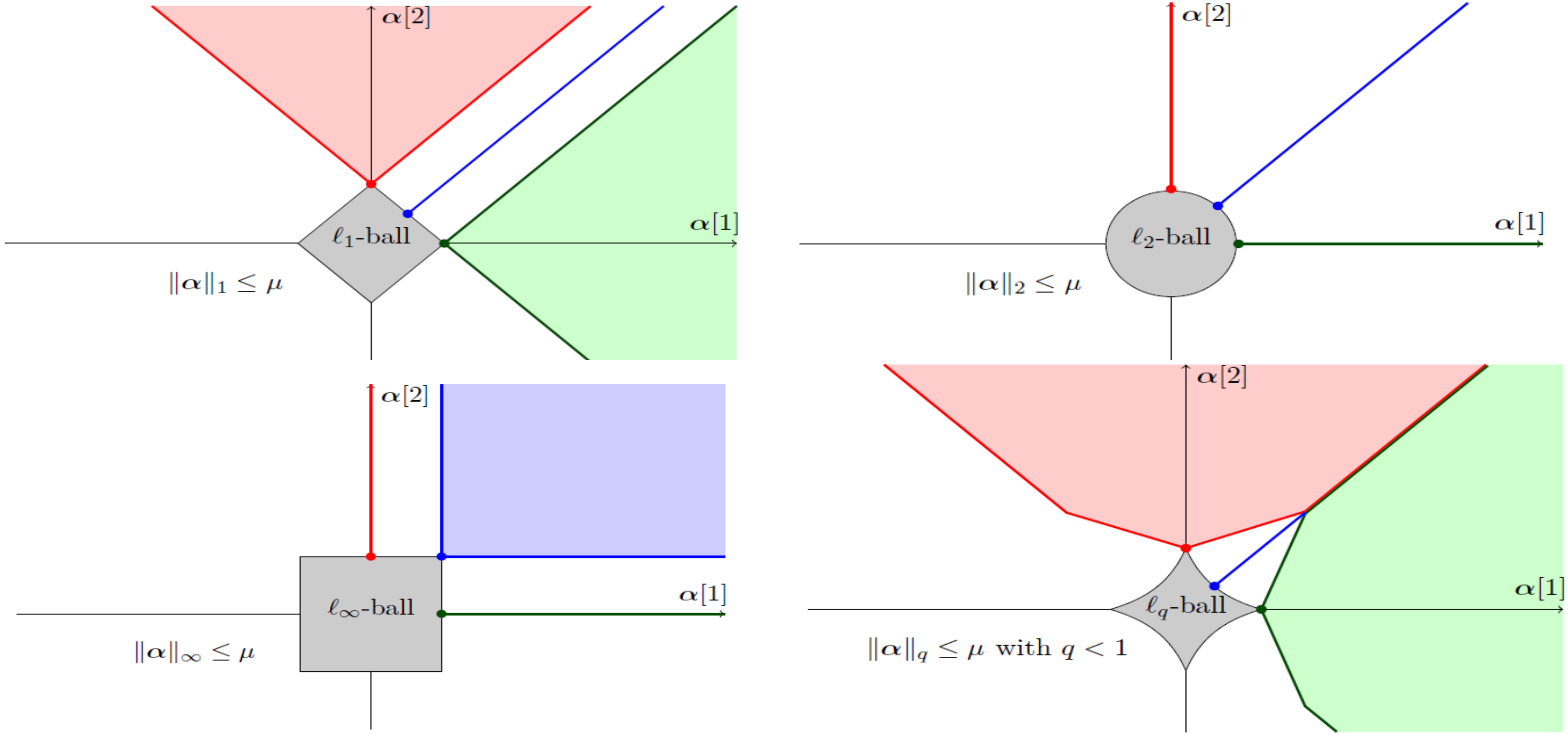
ψ induces sparsity in α. It can be selected as

- $\|\alpha\|_0 \triangleq \#\{i|\alpha(i) \neq 0\}$ (ℓ_0 “pseudo-norm”, NP-hard),
- $\|\alpha\|_1 \triangleq \sum_{i=1}^{k}|\alpha(i)|$ (ℓ_1 norm, convex),
- ……

Although (1) known to be NP-hard, the solution can be well approximated using numerous techniques, as long as α is sparse enough.

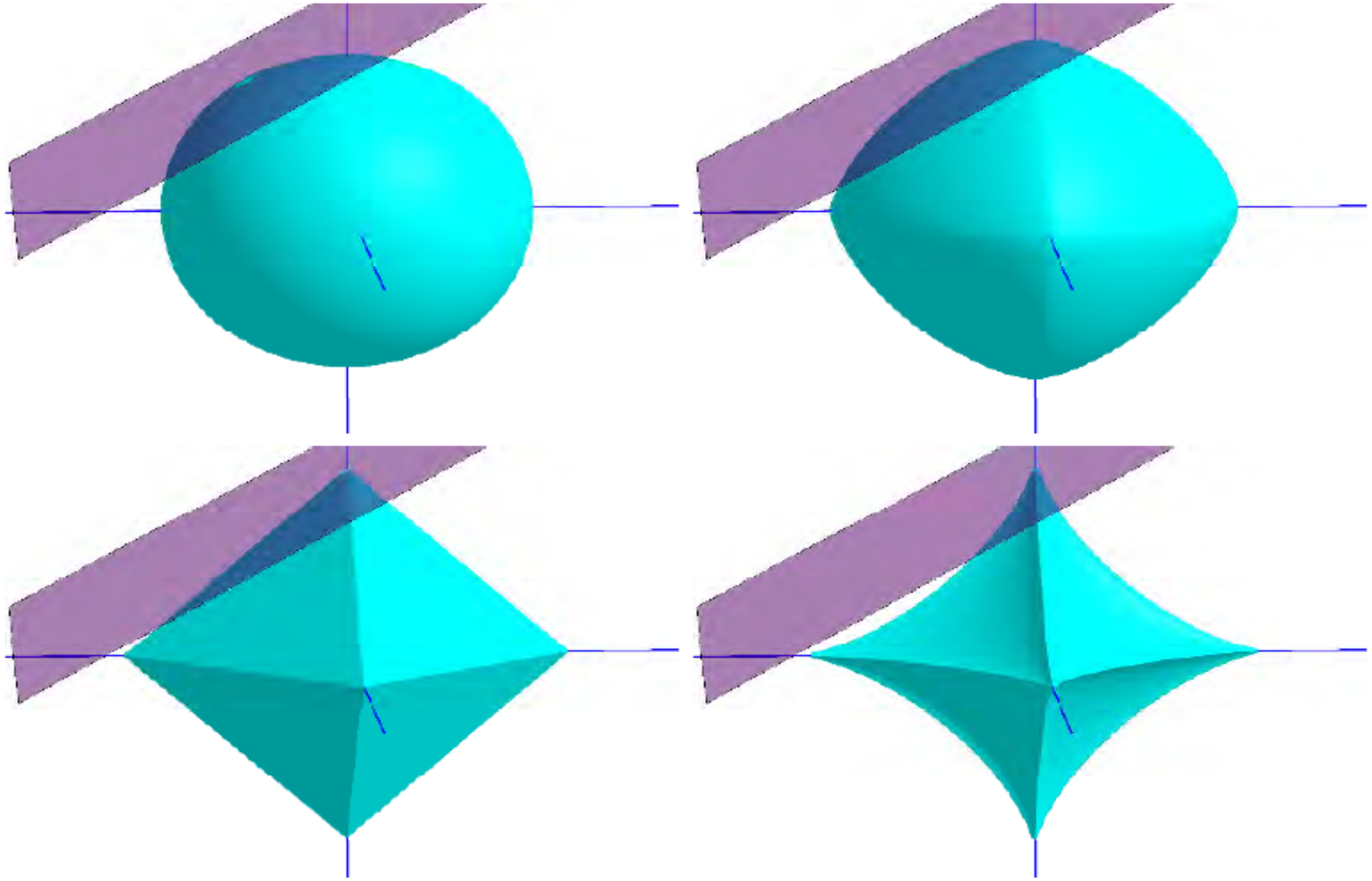
Why does the ℓ_1 norm induces sparsity?

Geometric illustration in 2D



Why does the ℓ_1 norm induces sparsity?

Geometric illustration in 3D



Application

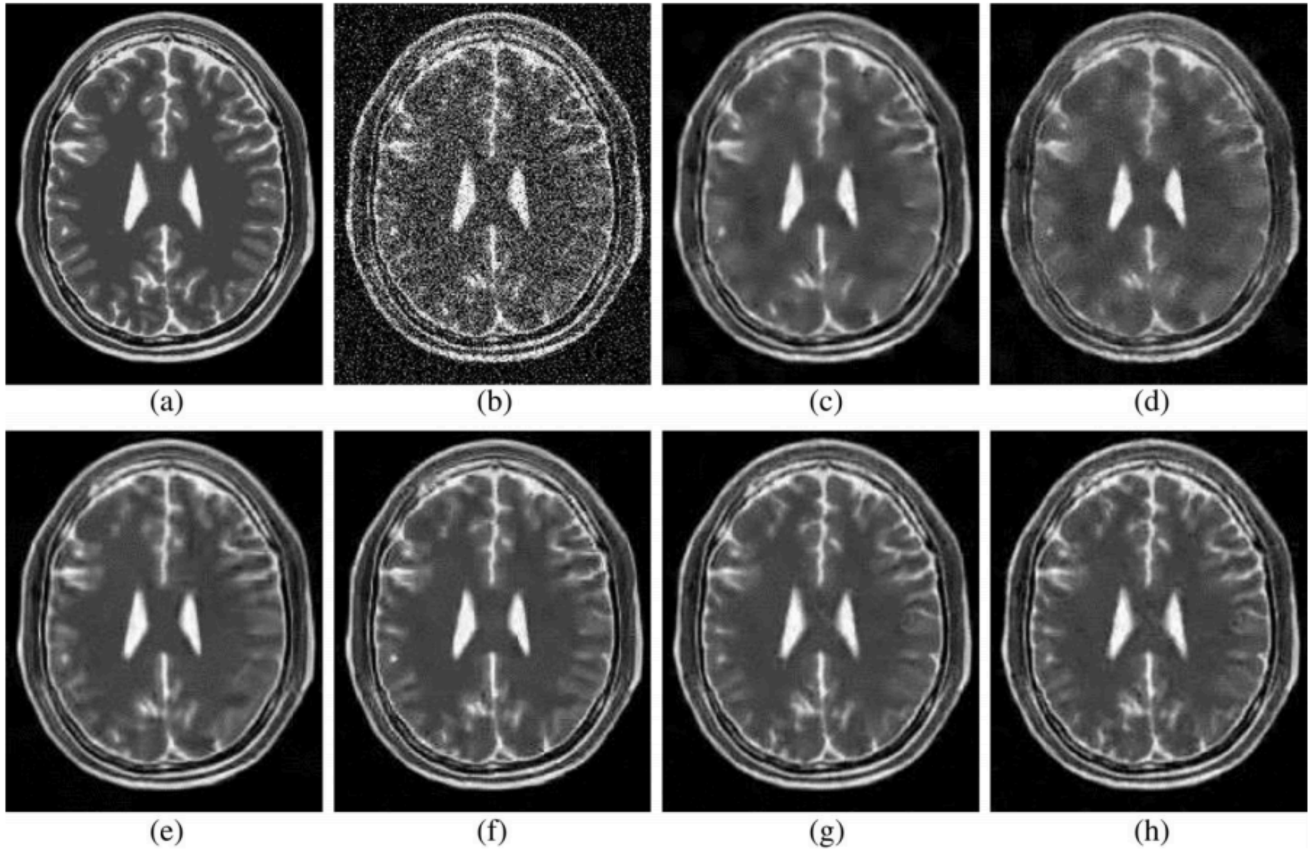


Fig. 4. Denoised results of T2-MR (slice 100) corrupted by the Gaussian noise. (a) Original image. (b) Noisy image (20% noise level). (c) 2-D K-SVD. (d) NLM. (e) BM3D. (f) VBM3D. (g) 3-D K-SVD. (h) 3-D DGSTR.

3-D DGSTR is 3-D Denoising with Group Sparsity and Temporal Regularization

Li S, Yin H, Fang L. Group-sparse representation with dictionary learning for medical image denoising and fusion[J]. IEEE Transactions on biomedical engineering, 2012, 59(12): 3450-3459.

Thank You!