PART 3
ADVANCED CIRCUIT ANALYSIS

CHAPTER 15
THE LAPLACE TRANSFORM

- *The important thing about a problem is not its solution, but the strength we gain in finding the solution.*

- The Laplace transform is significant for a number of reasons.
- First,it can be applied to a wider variety of inputs than phasor analysis.
- Second,it provides an easy way to solve circuit problems involving initial conditions, because it allows us to work with algebraic equations instead of differential equations.
- Third, the Laplace transform is capable of providing us, in one single operation, the total response of the circuit comprising both the natural and forced responses.

15.2 Definition of the Laplace Transform

Given a function $f(t)$, its Laplace transform, denoted by $F(s)$ or $\mathcal{L}[f(t)]$, is defined by

$$\boxed{\mathcal{L}[f(t)] = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt} \tag{15.1}$$

where s is a complex variable given by

$$s = \sigma + j\omega \tag{15.2}$$

Determine the Laplace transform of each of the following functions: (a) $u(t)$, (b) $e^{-at}u(t)$, $a \geq 0$, and (c) $\delta(t)$.

Solution:

(a) For the unit step function $u(t)$, shown in Fig. 15.2(a), the Laplace transform is

$$\mathcal{L}[u(t)] = \int_{0^-}^{\infty} 1e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_0^{\infty} = -\frac{1}{s}(0) + \frac{1}{s}(1) = \frac{1}{s} \tag{15.1.1}$$

(b) For the exponential function, shown in Fig. 15.2(b), the Laplace transform is

$$\mathcal{L}[e^{-at}u(t)] = \int_{0^-}^{\infty} e^{-at}e^{-st}\,dt = -\frac{1}{s+a}e^{-(s+a)t}\Big|_0^{\infty} = \frac{1}{s+a} \tag{15.1.2}$$

(c) For the unit impulse function, shown in Fig. 15.2(c),

$$\mathcal{L}[\delta(t)] = \int_{0^-}^{\infty} \delta(t)e^{-st}\,dt = e^{-0} = 1 \qquad \textbf{(15.1.3)}$$

since the impulse function $\delta(t)$ is zero everywhere except at $t = 0$. The sifting property in Eq. (7.33) has been applied in Eq. (15.1.3).

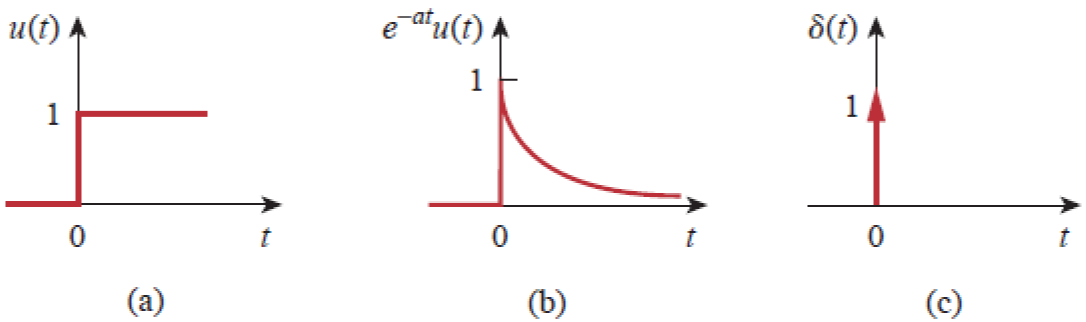


Figure 15.2

For Example 15.1: (a) unit step function, (b) exponential function, (c) unit impulse function.

Determine the Laplace transform of $f(t) = \sin \omega t\, u(t)$.

Solution:

Using Eq. (B.27) in addition to Eq. (15.1), we obtain the Laplace transform of the sine function as

$$F(s) = \mathcal{L}[\sin \omega t] = \int_0^\infty (\sin \omega t) e^{-st}\, dt = \int_0^\infty \left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right) e^{-st}\, dt$$

$$= \frac{1}{2j} \int_0^\infty \left(e^{-(s-j\omega)t} - e^{-(s+j\omega)t}\right) dt$$

$$= \frac{1}{2j}\left(\frac{1}{s - j\omega} - \frac{1}{s + j\omega}\right) = \frac{\omega}{s^2 + \omega^2}$$

Find the Laplace transform of $f(t) = 10 \cos \omega t\, u(t)$.

Answer: $10s/(s^2 + \omega^2)$.

TABLE 15.1

Properties of the Laplace transform.

Property	$f(t)$	$F(s)$
Linearity	$a_1f_1(t) + a_2f_2(t)$	$a_1F_1(s) + a_2F_2(s)$
Scaling	$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
Time shift	$f(t-a)u(t-a)$	$e^{-as}F(s)$
Frequency shift	$e^{-at}f(t)$	$F(s+a)$
Time differentiation	$\frac{df}{dt}$	$sF(s) - f(0^-)$
	$\frac{d^2f}{dt^2}$	$s^2F(s) - sf(0^-) - f'(0^-)$
	$\frac{d^3f}{dt^3}$	$s^3F(s) - s^2f(0^-) - sf'(0^-) - f''(0^-)$
	$\frac{d^nf}{dt^n}$	$s^nF(s) - s^{n-1}f(0^-) - s^{n-2}f'(0^-) - \cdots - f^{(n-1)}(0^-)$
Time integration	$\int_0^t f(t)dt$	$\frac{1}{s}F(s)$
Frequency differentiation	$tf(t)$	$-\frac{d}{ds}F(s)$
Frequency integration	$\frac{f(t)}{t}$	$\int_s^\infty F(s)ds$
Time periodicity	$f(t) = f(t+nT)$	$\frac{F_1(s)}{1-e^{-sT}}$
Initial value	$f(0)$	$\lim_{s\to\infty} sF(s)$
Final value	$f(\infty)$	$\lim_{s\to 0} sF(s)$
Convolution	$f_1(t) * f_2(t)$	$F_1(s)F_2(s)$

TABLE 15.2

Laplace transform pairs.*

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
te^{-at}	$\frac{1}{(s+a)^2}$
t^ne^{-at}	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t + \theta)$	$\frac{s\sin\theta + \omega\cos\theta}{s^2+\omega^2}$
$\cos(\omega t + \theta)$	$\frac{s\cos\theta - \omega\sin\theta}{s^2+\omega^2}$
$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

*Defined for $t \geq 0$; $f(t) = 0$, for $t < 0$.

The Inverse Laplace Transform

15.4.1 Simple Poles

Recall from Chapter 14 that a simple pole is a first-order pole. If $F(s)$ has only simple poles, then $D(s)$ becomes a product of factors, so that

$$F(s) = \frac{N(s)}{(s + p_1)(s + p_2)\cdots(s + p_n)} \tag{15.48}$$

where $s = -p_1, -p_2, \ldots, -p_n$ are the simple poles, and $p_i \neq p_j$ for all $i \neq j$ (i.e., the poles are distinct). Assuming that the degree of $N(s)$ is

less than the degree of $D(s)$, we use partial fraction expansion to decompose $F(s)$ in Eq. (15.48) as

$$F(s) = \frac{k_1}{s + p_1} + \frac{k_2}{s + p_2} + \cdots + \frac{k_n}{s + p_n} \tag{15.49}$$

$$\boxed{k_i = (s + p_i)F(s)\,\big|_{s=-p_i}}$$

15.4.2 Repeated Poles

Suppose $F(s)$ has n repeated poles at $s = -p$. Then we may represent $F(s)$ as

$$F(s) = \frac{k_n}{(s+p)^n} + \frac{k_{n-1}}{(s+p)^{n-1}} + \cdots + \frac{k_2}{(s+p)^2} + \frac{k_1}{s+p} + F_1(s) \tag{15.54}$$

The mth term becomes

$$k_{n-m} = \frac{1}{m!}\frac{d^m}{ds^m}[(s+p)^n F(s)]\,|_{s=-p}$$

where $F_1(s)$ is the remaining part of $F(s)$ that does not have a pole at $s = -p$. We determine the expansion coefficient k_n as

$$k_n = (s+p)^n F(s)\,|_{s=-p} \tag{15.55}$$

as we did above. To determine k_{n-1}, we multiply each term in Eq. (15.54) by $(s+p)^n$ and differentiate to get rid of k_n, then evaluate the result at $s = -p$ to get rid of the other coefficients except k_{n-1}. Thus, we obtain

$$k_{n-1} = \frac{d}{ds}[(s+p)^n F(s)]\,|_{s=-p} \tag{15.56}$$

Repeating this gives

$$k_{n-2} = \frac{1}{2!}\frac{d^2}{ds^2}[(s+p)^n F(s)]\,|_{s=-p} \tag{15.57}$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s+a)^n}\right] = \frac{t^{n-1}e^{-at}}{(n-1)!}u(t) \quad \textbf{(15.59)}$$

to each term on the right-hand side of Eq. (15.54) and obtain

$$\boxed{\begin{aligned} f(t) = \bigg(&k_1 e^{-pt} + k_2 t e^{-pt} + \frac{k_3}{2!}t^2 e^{-pt} \\ &+ \cdots + \frac{k_n}{(n-1)!}t^{n-1}e^{-pt}\bigg)u(t) + f_1(t) \end{aligned}} \quad \textbf{(15.60)}$$

3.Complex poles

$$F(s) = \frac{A_1 s + A_2}{s^2 + as + b} + F_1(s) \qquad \textbf{(15.61)}$$

where $F_1(s)$ is the remaining part of $F(s)$ that does not have this pair of complex poles. If we complete the square by letting

$$s^2 + as + b = s^2 + 2\alpha s + \alpha^2 + \beta^2 = (s + \alpha)^2 + \beta^2 \qquad \textbf{(15.62)}$$

and we also let

$$A_1 s + A_2 = A_1(s + \alpha) + B_1 \beta \qquad \textbf{(15.63)}$$

then Eq. (15.61) becomes

$$F(s) = \frac{A_1(s + \alpha)}{(s + \alpha)^2 + \beta^2} + \frac{B_1 \beta}{(s + \alpha)^2 + \beta^2} + F_1(s) \qquad \textbf{(15.64)}$$

From Table 15.2, the inverse transform is

$$\boxed{f(t) = (A_1 e^{-\alpha t} \cos \beta t + B_1 e^{-\alpha t} \sin \beta t) u(t) + f_1(t)} \qquad \textbf{(15.65)}$$

Use the Laplace transform to solve the differential equation

$$\frac{d^2v(t)}{dt^2} + 6\frac{dv(t)}{dt} + 8v(t) = 2u(t)$$

subject to $v(0) = 1, v'(0) = -2$.

Solution:

We take the Laplace transform of each term in the given differential equation and obtain

$$[s^2V(s) - sv(0) - v'(0)] + 6[sV(s) - v(0)] + 8V(s) = \frac{2}{s}$$

Solve for the response $y(t)$ in the following integrodifferential equation.

$$\frac{dy}{dt} + 5y(t) + 6\int_0^t y(\tau)\,d\tau = u(t), \qquad y(0) = 2$$

Solution:

Taking the Laplace transform of each term, we get

$$[sY(s) - y(0)] + 5Y(s) + \frac{6}{s}Y(s) = \frac{1}{s}$$

APPLICATIONS OF THE LAPLACE TRANSFORM

Steps in Applying the Laplace Transform:

1. Transform the circuit from the time domain to the *s*-domain.
2. Solve the circuit using nodal analysis, mesh analysis, source transformation, superposition, or any circuit analysis technique with which we are familiar.
3. Take the inverse transform of the solution and thus obtain the solution in the time domain.

For a resistor, the voltage-current relationship in the time domain is

$$v(t) = Ri(t) \qquad \textbf{(16.1)}$$

Taking the Laplace transform, we get

$$\boxed{V(s) = RI(s)} \qquad \textbf{(16.2)}$$

For an inductor,

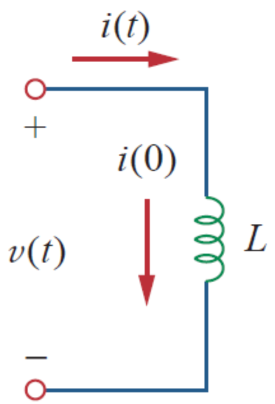
$$v(t) = L\frac{di(t)}{dt}$$

Taking the Laplace transform of both sides gives

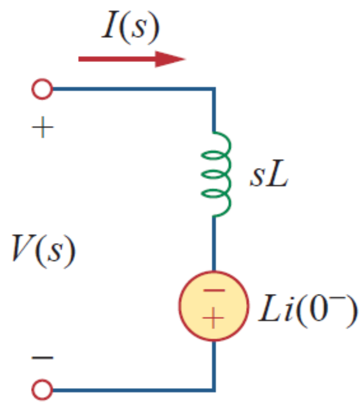
$$V(s) = L[sI(s) - i(0^-)] = sLI(s) - Li(0^-)$$

or

$$\boxed{I(s) = \frac{1}{sL}V(s) + \frac{i(0^-)}{s}}$$



(a)



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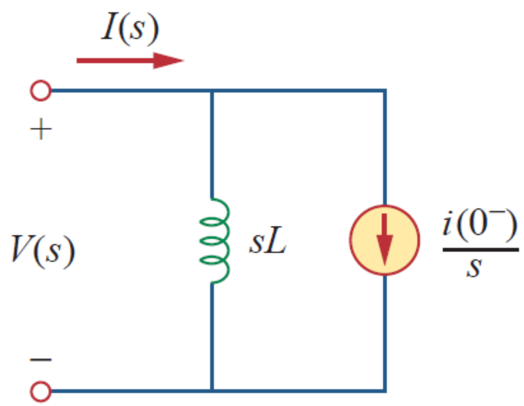


For a capacitor,

$$i(t) = C\frac{dv(t)}{dt}$$

which transforms into the s-domain as

$$I(s) = C[sV(s) - v(0^-)] = sCV(s) - Cv(0^-)$$

or

$$\boxed{V(s) = \frac{1}{sC}I(s) + \frac{v(0^-)}{s}}$$

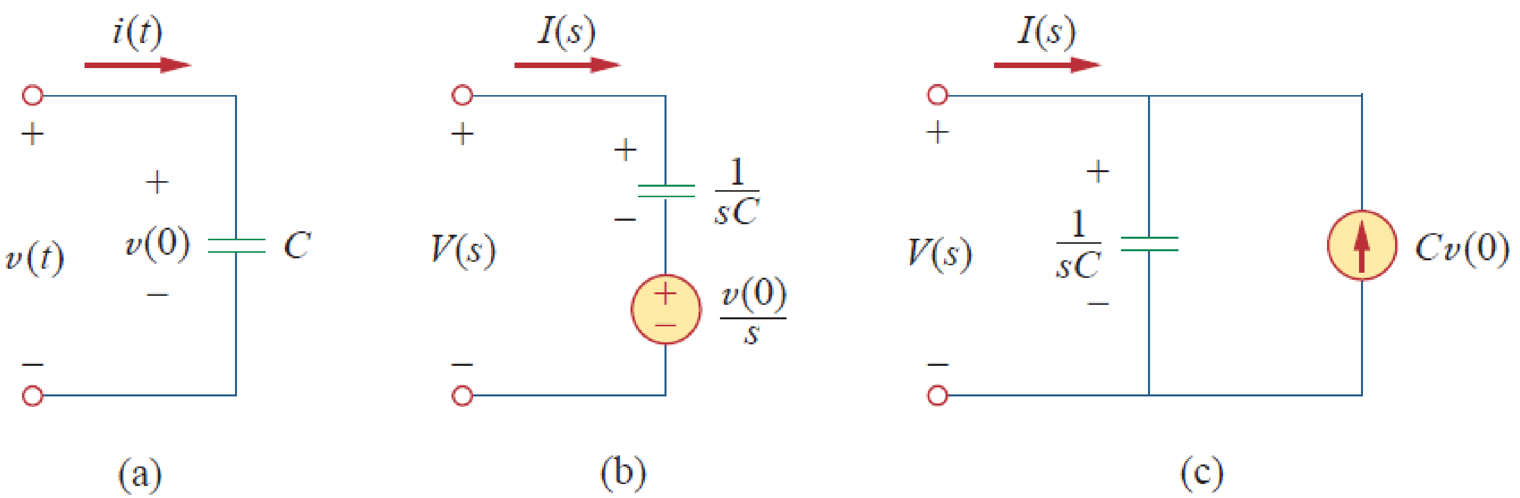


Figure 16.2
Representation of a capacitor: (a) time-domain, (b,c) *s*-domain equivalents.

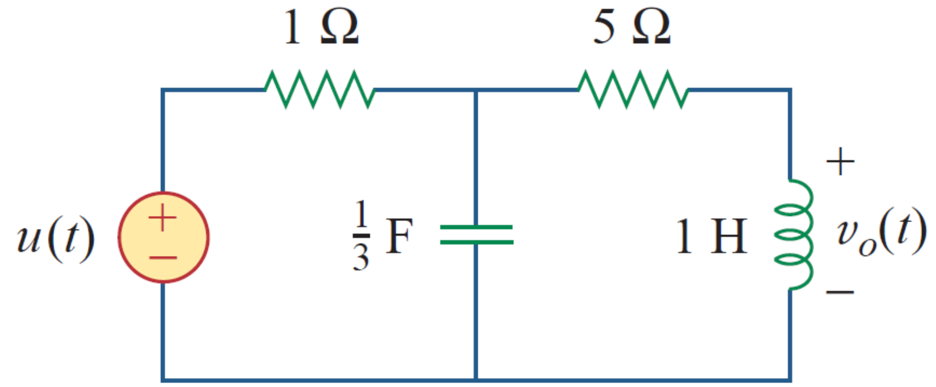


Figure 16.4
For Example 16.1.

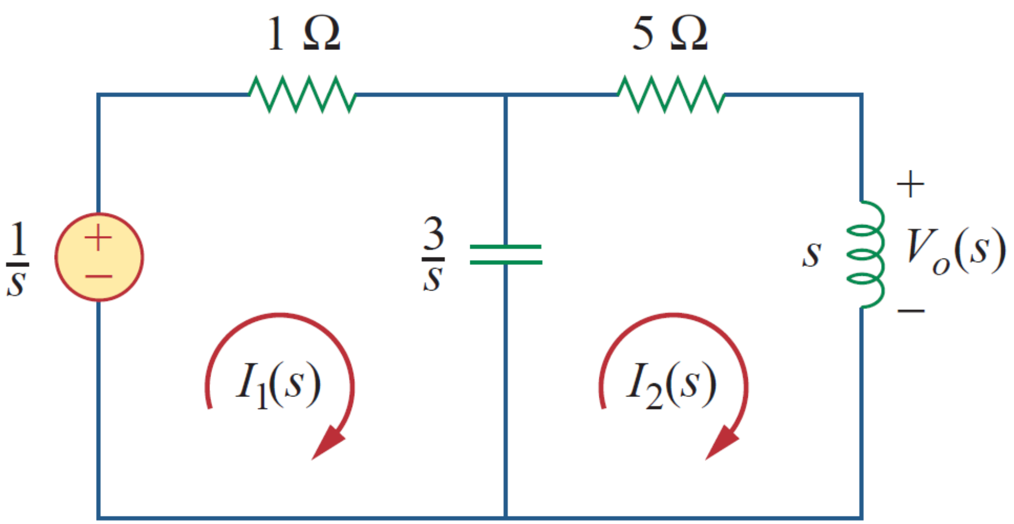
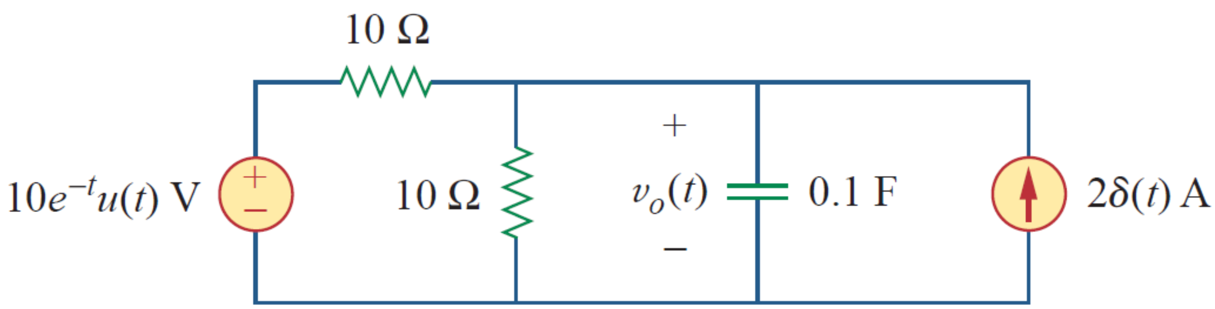
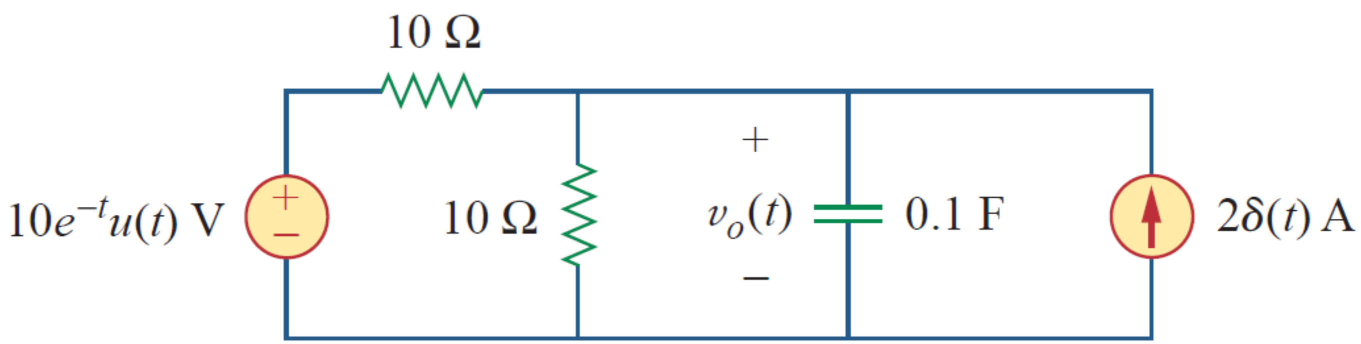


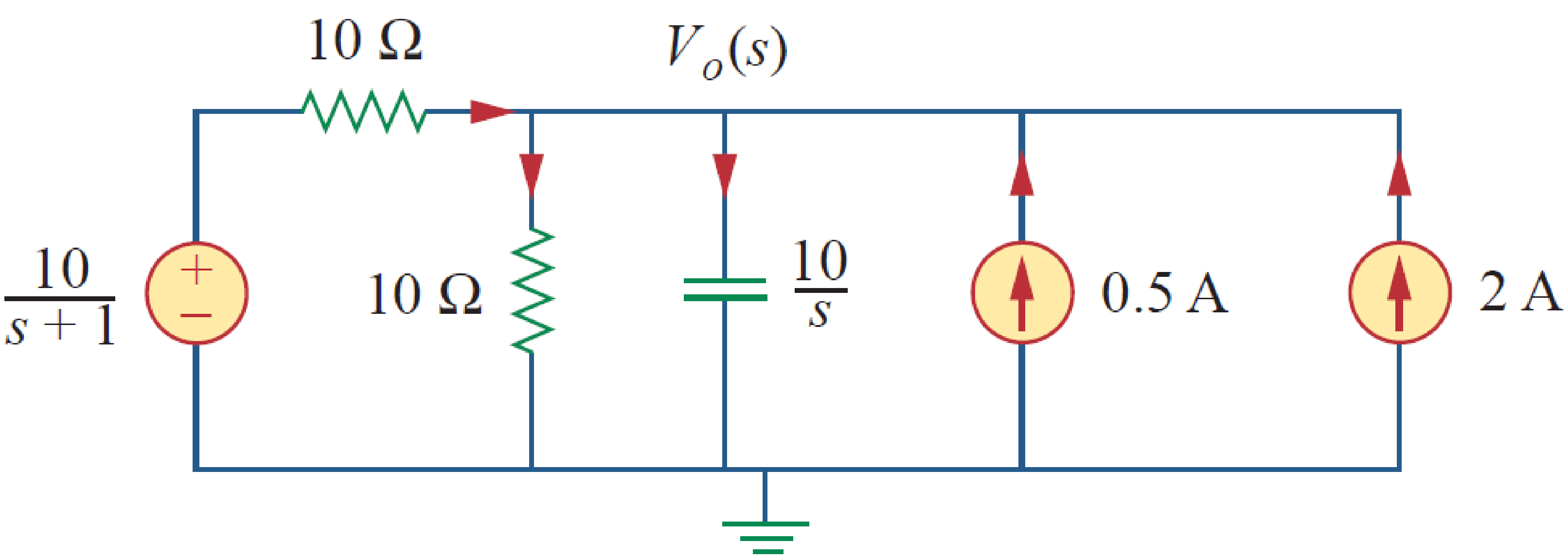
Figure 16.5
Mesh analysis of the frequency-domain equivalent of the same circuit.

Find $v_o(t)$ in the circuit of Fig. 16.7. Assume $v_o(0) = 5$ V.



Find $v_o(t)$ in the circuit of Fig. 16.7. Assume $v_o(0) = 5$ V.





Example 16.4

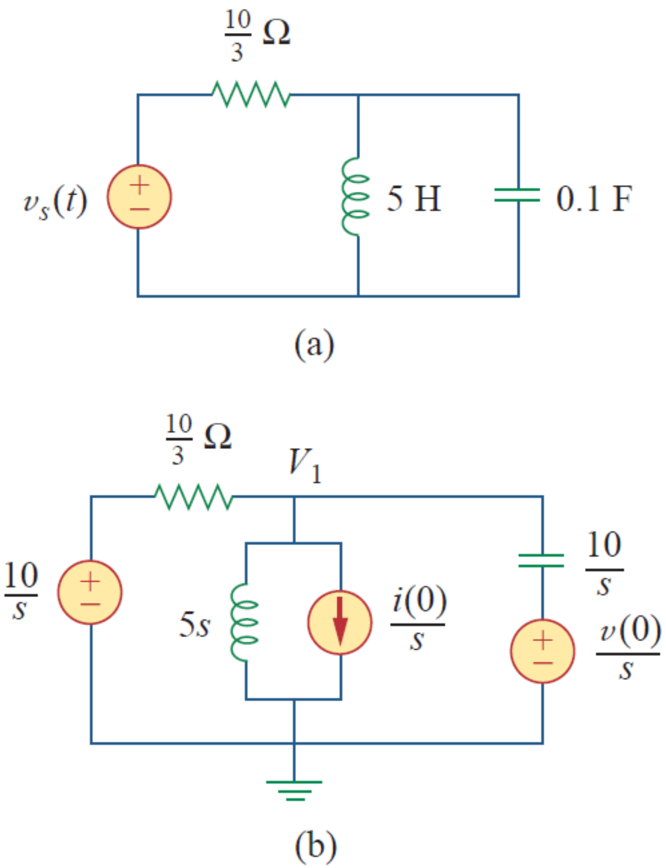


Figure 16.12
For Example 16.4.

$$\frac{V_1 - 10/s}{10/3} + \frac{V_1 - 0}{5s} + \frac{i(0)}{s} + \frac{V_1 - [v(0)/s]}{1/(0.1s)} = 0$$

The **transfer function** $H(s)$ is the ratio of the output response $Y(s)$ to the input excitation $X(s)$, assuming all initial conditions are zero.

Thus,

$$\boxed{H(s) = \frac{Y(s)}{X(s)}} \tag{16.15}$$

The transfer function depends on what we define as input and output. Since the input and output can be either current or voltage at any place in the circuit, there are four possible transfer functions:

$$H(s) = \text{Voltage gain} = \frac{V_o(s)}{V_i(s)} \tag{16.16a}$$

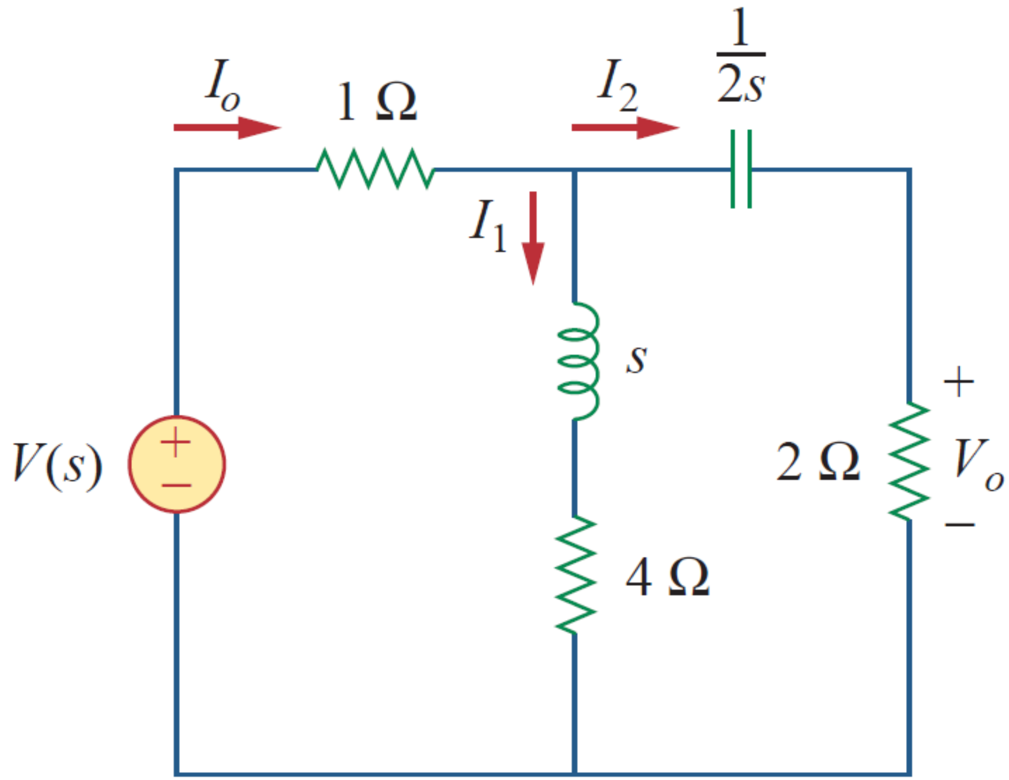
$$H(s) = \text{Current gain} = \frac{I_o(s)}{I_i(s)} \tag{16.16b}$$

$$H(s) = \text{Impedance} = \frac{V(s)}{I(s)} \tag{16.16c}$$

$$H(s) = \text{Admittance} = \frac{I(s)}{V(s)} \tag{16.16d}$$

Determine the transfer function $H(s) = V_o(s)/I_o(s)$ of the circuit

Example 16.8



STATE VARIABLE METHOD

Steps to Apply the State Variable Method to Circuit Analysis:

1. Select the inductor current i and capacitor voltage v as the state variables, making sure they are consistent with the passive sign convention.
2. Apply KCL and KVL to the circuit and obtain circuit variables (voltages and currents) in terms of the state variables. This should lead to a set of first-order differential equations necessary and sufficient to determine all state variables.
3. Obtain the output equation and put the final result in state-space representation.

Find the state-space representation of the circuit in Fig. 16.22. Determine the transfer function of the circuit when v_s is the input and i_x is the output. Take $R = 1\Omega$, $C = 0.25$ F, and $L = 0.5$ H.

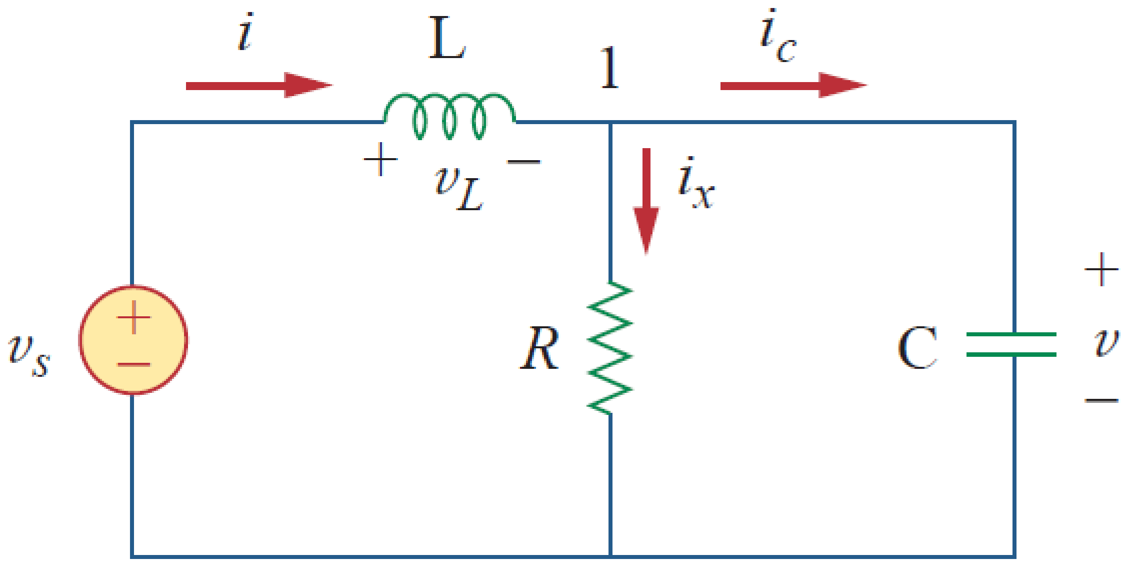


Figure 16.22
For Example 16.10.

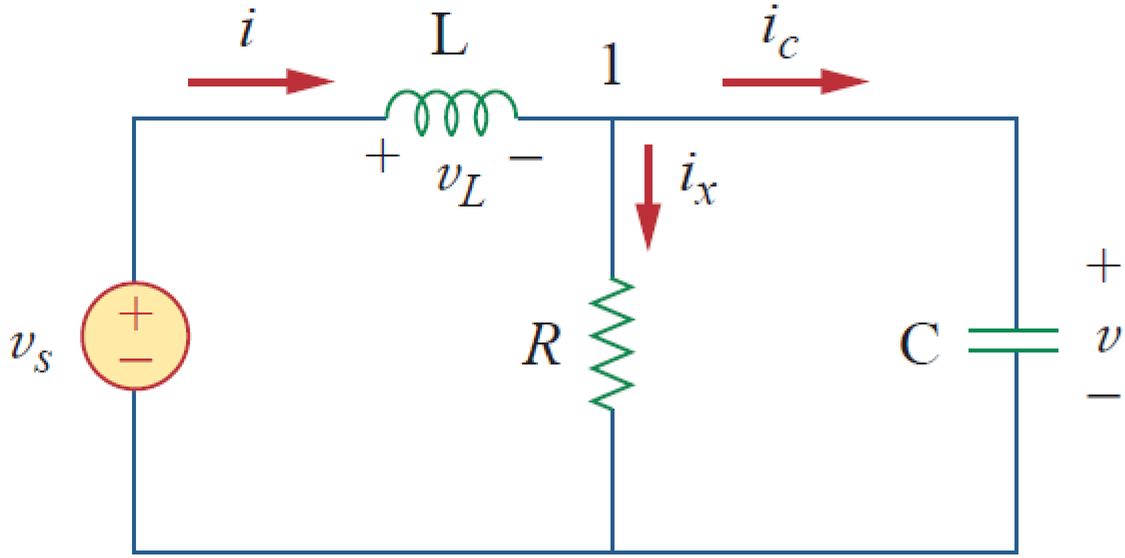


Figure 16.22

For Example 16.10.

$$\begin{bmatrix} \dot{v} \\ \dot{i} \end{bmatrix} = \begin{bmatrix} \frac{-1}{RC} & \frac{1}{C} \\ \frac{-1}{L} & 0 \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} v_s$$

$$i_x = \begin{bmatrix} \frac{1}{R} & 0 \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix}$$

Solution:

We select the inductor current i and capacitor voltage v as the state variables.

$$v_L = L\frac{di}{dt} \qquad \textbf{(16.10.1)}$$

$$i_C = C\frac{dv}{dt} \qquad \textbf{(16.10.2)}$$

Applying KCL at node 1 gives

$$i = i_x + i_C \quad \rightarrow \quad C\frac{dv}{dt} = i - \frac{v}{R}$$

or

$$\dot{v} = -\frac{v}{RC} + \frac{i}{C} \qquad \textbf{(16.10.3)}$$

since the same voltage v is across both R and C. Applying KVL around the outer loop yields

$$v_s = v_L + v \quad \rightarrow \quad L\frac{di}{dt} = -v + v_s$$

$$\dot{i} = -\frac{v}{L} + \frac{v_s}{L} \tag{16.10.4}$$

Equations (16.10.3) and (16.10.4) constitute the state equations. If we regard i_x as the output,

$$i_x = \frac{v}{R} \tag{16.10.5}$$

Putting Eqs. (16.10.3), (16.10.4), and (16.10.5) in the standard form leads to

$$\begin{bmatrix} \dot{v} \\ \dot{i} \end{bmatrix} = \begin{bmatrix} \frac{-1}{RC} & \frac{1}{C} \\ \frac{-1}{L} & 0 \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} v_s \qquad \textbf{(16.10.6a)}$$

$$i_x = \begin{bmatrix} \frac{1}{R} & 0 \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix} \qquad \textbf{(16.10.6b)}$$

If $R = 1$, $C = \frac{1}{4}$, and $L = \frac{1}{2}$, we obtain from Eq. (16.10.6) matrices

$$\mathbf{A} = \begin{bmatrix} \frac{-1}{RC} & \frac{1}{C} \\ \frac{-1}{L} & 0 \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ -2 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} \frac{1}{R} & 0 \end{bmatrix} = [1 \quad 0]$$

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -4 & 4 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} s+4 & -4 \\ 2 & s \end{bmatrix}$$

Taking the inverse of this gives

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{\text{adjoint of } \mathbf{A}}{\text{determinant of } \mathbf{A}} = \frac{\begin{bmatrix} s & 4 \\ -2 & s+4 \end{bmatrix}}{s^2 + 4s + 8}$$

Thus, the transfer function is given by

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \frac{[1 \quad 0]\begin{bmatrix} s & 4 \\ -2 & s+4 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix}}{s^2 + 4s + 8} = \frac{[1 \quad 0]\begin{bmatrix} 8 \\ 2s+8 \end{bmatrix}}{s^2 + 4s + 8}$$

$$= \frac{8}{s^2 + 4s + 8}$$

which is the same thing we would get by directly Laplace transforming the circuit and obtaining $\mathbf{H}(s) = I_x(s)/V_s(s)$. The real advantage of the state variable approach comes with multiple inputs and multiple outputs. In this case, we have one input v_s and one output i_x. In the next example, we will have two inputs and two outputs.

Obtain the state variable model for the circuit shown in Fig. 16.23. Let $R_1 = 1$, $R_2 = 2$, $C = 0.5$, and $L = 0.2$ and obtain the transfer function.

Practice Problem 16.10

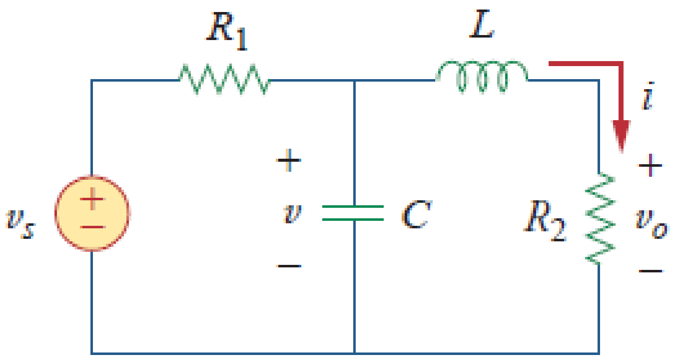


Figure 16.23
For Practice Prob. 16.10.

Consider the circuit in Fig. 16.24, which may be regarded as a two-input, two-output system. Determine the state variable model and find the transfer function of the system.

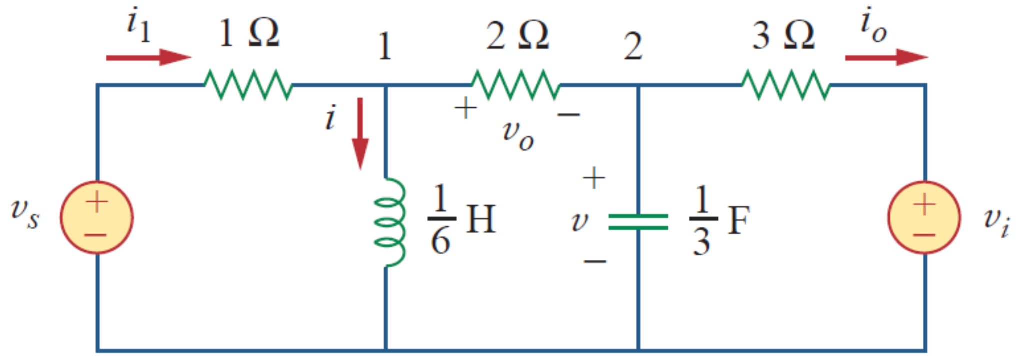


Figure 16.24
For Example 16.11.

$$\begin{bmatrix} \dot{v} \\ \dot{i} \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} v_s \\ v_i \end{bmatrix}$$

$$\begin{bmatrix} v_o \\ i_o \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & -\frac{2}{3} \\ \frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix} + \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} v_s \\ v_i \end{bmatrix}$$